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THE PHYSICAL ORIGIN OF FIDELITY ENHANCEMENT IN THE TELEPORTATION PROTOCOL OF AN ARBITRARY TWO-QUBIT STATE UNDER AMPLITUDE-DAMPING NOISE ENVIRONMENT

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Abstracts. A new analytical expression for the average quantum fidelity of a teleportation protocol for an arbitrary two-qubit state transmitted through an amplitude-damping noise environment has been established. This expression shows that the average fidelity depends not only on the purity measure and the bipartite entanglement measure of each parallel state forming the quantum channel, but also on the multipartite entanglement measure. This entanglement is generated between the qubits of each parallel state when they interact independently with the amplitude-damping environment. Based on this analytical relation, the physical mechanism for enhancing the teleportation protocol's average quantum fidelity under two distinct amplitude-damping noise scenarios has been elucidated.

Keywords: amplitude-damping noise, optimal averaged fidelity, entanglement measure, purity measure, multipartite entanglement.

1. Introduction

In practice, any quantum system interacts with its environment, which transforms the pure quantum state of the system into a mixed state [1]. In [2], Popescu showed that a mixed state used as a quantum channel can sometimes be more efficient for quantum state teleportation; namely, the quantum fidelity is larger than the best value obtainable by classical teleportation. The quantum fidelity of a teleportation protocol depends not only on the amount of entanglement present in the mixed state but also on its classical characteristics. Therefore, we regard the degree of mixedness as an independent physical criterion that can affect the performance of quantum entanglement-based applications, particularly quantum teleportation.

In 2006, Ye Yeo [3] investigated the teleportation of an arbitrary two-qubit state through a quantum channel given by a mixed state of four qubits, which is equivalent to a generalized depolarizing bi-channel. For the teleportation protocol to achieve an average quantum fidelity exceeding the classical threshold, the generalized singlet fraction of the quantum channel must satisfy a specific condition. When this condition is not met, i.e., when the generalized singlet fraction falls below the threshold (i.e., when the generalized singlet fraction falls below the threshold), faithful teleportation of the quantum state is no longer possible. Cyclic quantum teleportation of two-qubit entangled states using quantum channels based on six-qubit cluster states and six-qubit entangled states was explored in [4]. In [5], the authors measured and tracked the evolution of entanglement and fidelity of prepared two-qubit states on IBM's 127-qubit quantum processor, where one of the qubits was teleported across a sequence of 19 physical qubits.

In [6], Ye Yeo considered the teleportation of an arbitrary two-qubit state using a family of four-qubit mixed states as quantum channels. This framework showed that the average fidelity of the two-qubit teleportation protocol can be enhanced via dissipative interaction with a localized environment. However, the underlying physical mechanism responsible for this enhancement was not identified. The influence of noise on two classes of quantum channels, which can be either parallel Bell pairs or inseparable channels with genuine multipartite entanglement, was investigated in [7] in the context of arbitrary two-qubit state teleportation. The study demonstrated that when the sender's qubits, or both the sender's and receiver's qubits, are subjected to noise, the protocol fidelity using genuine multipartite entangled channels is, at best, equal to that obtained with parallel Bell pairs. This result highlights that multipartite entanglement does not necessarily guarantee improved teleportation performance under noise. The study of teleporting an entangled state of two qubits in [8] showed that the quantum entanglement of the resulting state strongly depends on the source state, the initial quantum entanglement of the teleported state, and the parameters of the noisy channel. In contrast, the average fidelity is only governed by the parameters of the source state and the noisy channels. Furthermore, in [9], quantum teleportation in both single-qubit and two-qubit scenarios was studied using a two-qubit Heisenberg XYZ spin chain model subject to atomic dipole interactions and the KSEWA interaction. The results demonstrated that by tuning the system parameters, both the average teleportation fidelity and the precision of encoded phase estimation for the teleported two-qubit state can be significantly improved.

Recently, the authors of [10] investigated quantum teleportation under a generalized noisy environment that continuously connects amplitude-damping and phase-damping noise, simultaneously affecting both qubits of the quantum channel shared between the sender and the receiver. Their study shows that dynamically generated multipartite entanglement can serve as an additional resource for quantum teleportation. Furthermore, the work sheds light on the nature of the processes and the types of entanglement generated, depending on their capacity to support quantum state transfer.

However, previous research has not analytically established complementary relationships among characteristic quantities of the quantum channel in the context of quantum teleportation. Specifically, the classicality of the mixed state through the purity measure of the quantum channel, the nonlocality through the entanglement measure of the quantum channel, the multipartite entanglement generated between the channel qubits and their environments, and the efficiency of the quantum channel represented by the average teleportation fidelity have yet to be rigorously clarified for the case of teleporting an arbitrary two-qubit state under amplitude-damping noise.

In this work, we investigate the characteristic quantities in the context of quantum teleportation of an arbitrary two-qubit state under amplitude-damping noise, where the initial quantum channel is constructed from a pair of Bell-like states. The qubits constituting the quantum channel are independently affected by local amplitude-damping environments. We derive analytical expressions for the average teleportation fidelity as functions of the purity and entanglement of each component state in the parallel pair forming the quantum channel, as well as the multipartite entanglement generated between the channel qubits and their respective environments. Based on these relations, we show the physical mechanisms underlying the enhancement of average fidelity in two specific scenarios:

i) In the symmetric noise scenario, where the noise strength acting on each component state of the parallel pair is the same ($p_1 = p_2 = p$), the average teleportation fidelity depends solely on the entanglement of each Bell-like component state forming the channel. Consequently, improving the teleportation fidelity corresponds to enhancing the entanglement of each component state. When the free parameter defining the initial states of the channel is appropriately matched to the noise strength, the average fidelity exceeds that obtained when using maximally entangled Bell states across the entire noise regime.

ii) In the asymmetric noise scenario, where the two Bell-like states composing the quantum channel are subjected to different noise strengths $(p_1 \neq p_2)$, the average teleportation fidelity depends not only on the entanglement but also on the purity of each component state, as well as on the multipartite entanglement generated between the channel qubits and their respective environments. Hence, the physical mechanism for enhancing the average fidelity in this case involves simultaneous optimization of all these contributing quantities. Again, by tuning the free parameter of the initial states to the corresponding noise strengths, the average fidelity can be consistently improved beyond that achievable using maximally entangled Bell states across the full range of noise intensities. Notably, in this scenario, the improvement is primarily attributed to increased purity of the component states comprising the quantum channel.

2. Content

2.1. The characteristic quantities of the quantum channel in the teleportation of an arbitrary two-qubit state under the influence of an amplitude-damping noise environment

Suppose the sender, Alice, wishes to transmit an arbitrary two-qubit state to the receiver, Bob. The state is assumed to be completely unknown to both parties and, in the general case, can be expressed as follows:

$$|\psi\rangle_{S_1S_2} = \left(\alpha_{00}|00\rangle + \alpha_{01}e^{i\varphi_1}|01\rangle + \alpha_{10}e^{i\varphi_2}|10\rangle + \alpha_{11}e^{i\varphi_3}|11\rangle\right)_{S_1S_2}. \quad (1)$$

Here, the coefficients are real $\alpha_{ij}, i, j = 0,1$ and satisfy the normalization condition $\sum_{i,j=0}^{1} \alpha_{ij}^2 = 1$, while the phases of the state to be teleported satisfy an additional constraint $0 \le \varphi_i \le 2\pi,\ i = 1,2,3.$ In teleporting an arbitrary two-qubit state through an ideal (i.e., noiseless) environment, the protocol can be considered perfect, meaning that both the success probability and the fidelity reach unity. In such ideal scenarios, the quantum channel employed can either consist of two parallel Bell states or a genuine four-qubit entangled state. The study by the authors in [7] demonstrated that both types of quantum channels, the pair of parallel Bell states and the genuine four-qubit entangled state, yield comparable fidelities. However, due to the greater technological complexity of generating genuine four-qubit entangled states, we use a pair of Bell-like entangled states as our parallel channel. Once generated, these entangled states are distributed to the participants of the teleportation protocol. During distribution, the states inevitably interact with the environment, leading to degradation in their entanglement and/or coherence and purity. To provide a more comprehensive analytical evaluation of the physical mechanism underlying the influence of mixedness (i.e., classicality) of the parallel Bell-like states due to environmental interactions, we analyze the behavior of the purity (as the complement of mixedness) and the degree of quantum nonlocality via the concurrence. These are considered in relation to the average quantum fidelity, which serves as a characteristic measure of the nonclassical applicability of the quantum channel in the context of teleporting arbitrary two-qubit states. Therefore, we select the initial state of each of the two Bell-like states composing the quantum channel to be an entangled two-qubit Bell-type state. Since the two parallel Bell-like states are identical and symmetrically shared between the sender and receiver, the analysis of channel characteristics in the teleportation context can be effectively carried out using only one of them. The Bell-like entangled state takes the following form:

$$\left|Q(\theta)\right\rangle_{AB} = \left(\cos\theta\left|00\right\rangle + \sin\theta\left|11\right\rangle\right)_{AB}. \tag{2}$$

The characteristic quantities of the states in Eq. (2) depend on the parameter θ, which is an undetermined value and hence referred to as a free parameter. The domain of this free parameter is chosen to satisfy the condition $0 \le \theta \le \pi/4$ [11]. For instance, a key characteristic of the state in Eq. (2) is its entanglement, which also represents the initial entanglement of each Bell-like state in the pair of parallel states used as the quantum channel in the teleportation protocol. If the concurrence is used as the entanglement measure, this quantity depends on the free parameter according to the expression $C(\theta) = \sin(2\theta)$. This shows that the larger the value θ, the stronger the initial entanglement of the quantum channel. In particular, when $\theta = \pi/4$ the state in Eq. (2) becomes a maximally entangled Bell state. When the qubits of the quantum state in Eq. (2) interact with an external environment, we assume that qubits A and B interact independently with their respective local environments, each modeled as an amplitude-damping noise channel. Therefore, the initial state of each Bell-like component of the parallel quantum channel and its corresponding environment is considered a product state. Moreover, we assume that each environment is initially in a vacuum state. Under these assumptions, the total initial state of each Bell-like entangled pair and its associated environment takes the form

$$\left|Q_0\left(\theta\right)\right\rangle_{ABE_1E_2} = \left(\cos\theta\left|00\right\rangle + \sin\theta\left|11\right\rangle\right)_{AB} \otimes \left|00\right\rangle_{E_1E_2}. \tag{3}$$

Here, the environment is modeled as a two-level system. The first level, or ground state, corresponds to the vacuum state of the environment. The environment is considered a single-mode quantized electromagnetic field inside a cavity, in which case the second level represents the first excited state of the monochromatic field. Under this assumption, the environment can be effectively treated as a qubit, which is thus referred to as an environmental qubit. Specifically, the environmental qubit E1 is associated with qubit A, and the environmental qubit E2 is associated with qubit B. Since the noise model under consideration is amplitude-damping, each qubit of the quantum channel is assumed to interact independently with its corresponding environmental qubit. The interaction between a system qubit and its local environment follows the amplitude-damping mechanism, which is governed by the following transformation rules:

$$\left|0\right\rangle_q\left|0\right\rangle_E \to \left|0\right\rangle_q\left|0\right\rangle_E, \tag{4}$$

$$\left|1\right\rangle_q\left|0\right\rangle_E \to \sqrt{1-p}\left|1\right\rangle_q\left|0\right\rangle_E + \sqrt{p}\left|0\right\rangle_q\left|1\right\rangle_E. \tag{5}$$

Here, p denotes the noise strength representing the influence of the environment on each qubit. Consequently, the composite state of each parallel component of the quantum channel and its corresponding environment forms a pure four-qubit quantum state. The initial state of each parallel component of the quantum channel, taken as a Bell-like state, can be expressed as

$$\begin{aligned}\left|Q\left(\theta,p_1,p_2\right)\right\rangle_{AE_1BE_2} &= \cos\theta\left|0000\right\rangle_{AE_1BE_2} + \sin\theta\Big[\sqrt{\left(1-p_1\right)\left(1-p_2\right)}\left|1010\right\rangle \\ &+\sqrt{\left(1-p_1\right)p_2}\left|1001\right\rangle + \sqrt{p_1\left(1-p_2\right)}\left|0110\right\rangle + \sqrt{p_1p_2}\left|0101\right\rangle\Big]_{AE_1BE_2}.\end{aligned} \tag{6}$$

When each qubit of the parallel components of the quantum channel interacts independently with its corresponding environmental qubit, the resulting composite system evolves into a pure four-qubit quantum state, referred to as state (6). This composite state can exhibit genuine four-qubit entanglement [12]. Previous studies, such as [10], investigating the teleportation of an arbitrary single-qubit state through an amplitude-damping noisy channel, have shown that dynamically generated multipartite correlations can serve as an additional resource for quantum teleportation. These correlations illuminate the processes' characteristics and the types of quantum correlations generated, depending on their ability to support the quantum teleportation protocol. However, the role of such multipartite correlations in teleporting an arbitrary two-qubit state through an amplitude-damping noisy channel, where the quantum channel consists of a pair of parallel Bell-like states, remains an open question. Therefore, this work focuses on the quantum entanglement generated among the four qubits in state (6). This entanglement is quantified using the 4-tangle τ_4, an entanglement measure specifically designed for pure states of an even number of qubits [13]. The 4-tangle is expressed as follows:

$$\tau_4 = \left| {}_{AE_1BE_2}\langle Q(\theta, p_1, p_2)| \sigma_y^{\otimes 4} |Q^*(\theta, p_1, p_2)\rangle_{AE_1BE_2} \right|^2, \tag{7}$$

in there $|Q^*(\theta, p_1, p_2)\rangle$ is to take the complex conjugate of $|Q(\theta, p_1, p_2)\rangle$. At that time

$$\tau_4 = 16 p_1 (1-p_2)(1-p_1) p_2 \sin^4\theta. \tag{8}$$

The state of each pair of parallel qubits in the quantum channel, initially prepared in the Bell-like state given by Eq. (2), when subjected to interaction with an amplitude-damping noisy environment, can be described by the reduced density matrix of the following form:

$$\begin{aligned}\rho_{AB}(\theta, p_1, p_2) &= Tr_{E_1E_2}\left(|Q(\theta, p_1, p_2)\rangle_{AE_1BE_2}\langle Q(\theta, p_1, p_2)|\right) \\ &= \left(\cos^2\theta + p_1p_2\sin^2\theta\right)|00\rangle_{AB}\langle 00| + \sin\theta\cos\theta\sqrt{(1-p_1)(1-p_2)}\left(|00\rangle_{AB}\langle 11| + |11\rangle_{AB}\langle 00|\right) \\ &+ p_1(1-p_2)\sin^2\theta|01\rangle_{AB}\langle 01| + (1-p_1)p_2\sin^2\theta|10\rangle_{AB}\langle 10| + (1-p_1)(1-p_2)\sin^2\theta|11\rangle_{AB}\langle 11|.\end{aligned} \tag{9}$$

The reduced density matrix (9) describes the mixed state of the two qubits, A and B, whose initial state is given by Eq. (2). This density matrix is of rank 4.

To characterize the classical nature of the quantum channel, in this work, we focus on the classical correlations emerging in the mixed state due to the interaction of the channel's qubits with the environment. Since noise tends to increase the mixedness of the quantum channel, mixedness arises as an intuitive parameter for understanding the channel's decoherence. It is well known that decoherence adversely affects the quantum information encoded in a quantum state, which can be quantified by the purity of that state. To effectively describe the role of decoherence in erasing quantum information, it is necessary to quantify purity or its complementary attribute, the mixedness of the state. For any quantum state, common interactions with the environment or decoherence influence its purity. Noise induces mixing in the quantum system, leading to information loss; therefore, understanding its characteristics is essential in quantum information protocols. Mixedness, which essentially represents the disorder within the system, can be quantified by entropic functions such as linear entropy or von Neumann entropy $S(\rho) = -Tr(\rho\log\rho)$ alternatively, by the purity of the quantum state. In this paper, we choose the quantum purity measure, defined by the expression. $P(\rho) = Tr(\rho^2)$. The purity measure attains its maximum value when the state is a rank-one projection, i.e., a pure state. Conversely, its minimum value equals the inverse of the dimension of the Hilbert space in which the state resides. The purity of the quantum state described by the reduced density matrix (9) is given by

$$\begin{aligned}P_{AB} = Tr\left[\rho_{AB}^2(p_1, p_2, \theta)\right] &= 1 - 2(p_1 + p_2 - 2p_1p_2)\cos^2\theta\sin^2\theta \\ &- 2\left[p_1(1-p_1) + p_2(1-p_2) - 2p_1(1-p_1)p_2(1-p_2)\right]\sin^4\theta.\end{aligned} \tag{10}$$

The purity of the quantum state described by the reduced density matrix (9) depends not only on the noise strength of the independent environments acting on the qubits but also on the free parameter of the initial Bell-like state (2) of one of the two parallel states forming the quantum channel, namely state (9).

Similarly to the measure of the quantum channel's purity, there are several entanglement measures applicable when the channel is a two-qubit quantum state, specifically, the von Neumann entropy, concurrence, and negativity. The von Neumann entropy is well-suited for pure quantum states; however, in our case, the state (9) becomes strongly mixed due to environmental interactions, so the von Neumann entropy does not provide an adequate description. Both concurrence and negativity are appropriate for mixed two-qubit states, although some states may exhibit stronger entanglement according to one measure while appearing weaker according to the other. In this paper, we focus on elucidating the relationship among the key quantities that characterize the quantum channel in the context of quantum teleportation, such as the channel's purity, its quantum entanglement, and the average quantum fidelity. Therefore, we employ Wootters' concurrence [14] to quantify the entanglement of the quantum state (9). In the general case, the concurrence is defined by the following formula

$$C_{AB} = \max\left\{0, \lambda_1 - \lambda_2 - \lambda_3 - \lambda_4\right\}, \tag{11}$$

where the $\lambda_i, i = 1 \div 4$ are arranged in non-increasing order of the eigenvalues of the matrix $\sqrt{\sqrt{\rho_{AB}}\tilde{\rho}_{AB}\sqrt{\rho_{AB}}}$ here $\tilde{\rho}_{AB} = \left(\sigma_y \otimes \sigma_y\right)\rho^*_{AB}\left(\sigma_y \otimes \sigma_y\right)$, the symbol '*' denotes the conjugate transpose of the matrix ρ_{AB}, $\sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$. In the case where the matrix representing the state of the system is of the form X, the entanglement measure, known as concurrence, is computed according to the following expression:

$$C = \max\left\{2\left(|\rho_{14}| - \sqrt{\rho_{22}\rho_{33}}\right), 2\left(|\rho_{23}| - \sqrt{\rho_{11}\rho_{44}}\right), 0\right\}. \tag{12}$$

Since the matrices representing the state of the quantum state (9) are of the form X, the concurrence for this state can be readily obtained, with the corresponding expressions given by

$$C_{AB} = \sqrt{(1-p_1)(1-p_2)}\sin(2\theta) - 2\sqrt{p_1(1-p_1)p_2(1-p_2)}\sin^2\theta. \tag{13}$$

2.2. Optimization of the average quantum fidelity in the teleportation protocol of an arbitrary two-qubit state through an amplitude-damping noisy channel

2.2.1. Dependence of average quantum fidelity on characteristic quantities in the context of teleportation

The initial quantum channel shared between the sender, Alice, and the receiver, Bob, before introducing noise on the qubits, consists of a pair of parallel Bell-like states, also known as a pair of separable states

$$\left|Q(\theta_1,\theta_2)\right\rangle_{A_1B_1A_2B_2}=\left|Q(\theta_1)\right\rangle_{A_1B_1}\otimes\left|Q(\theta_2)\right\rangle_{A_2B_2}, \qquad (14)$$

where the Bell-like states are explicitly given by the following expressions

$$\left|Q(\theta_1)\right\rangle_{A_1B_1}=\left(\cos\theta_1\left|00\right\rangle+\sin\theta_1\left|11\right\rangle\right)_{A_1B_1} \qquad (15)$$

and

$$\left|Q(\theta_2)\right\rangle_{A_2B_2}=\left(\cos\theta_2\left|00\right\rangle+\sin\theta_2\left|11\right\rangle\right)_{A_2B_2}. \qquad (16)$$

In the teleportation protocol of an arbitrary two-qubit state, given by the state (1), the qubits of the quantum channel (14) are distributed as follows: qubits A_1 and A_2 belong to the sender, Alice, while qubits B_1 and B_2 belong to the receiver, Bob. To successfully teleport the arbitrary two-qubit state using the standard protocol, the following steps are performed: Alice performs a Bell-state measurement on the pair of qubits S_1 and A_1, obtaining a measurement outcome encoded as $\{i,j\}$; similarly, she performs a Bell-state measurement on the pair S_2 and A_2, with the outcome encoded as $\{k,l\}$. These measurement results are then communicated to Bob via a classical channel.

Upon receiving the measurement results from the sender Alice, the receiver Bob applies appropriate recovery operations on qubits B_1 and B_2 as follows: For qubit B_1, the recovery operator applied by Bob is given by $R_{B_1}=Z^i.X^j$; for qubit B_2, the recovery operator is $R_{B_2}=Z^k.X^l$, where X and Z are the Pauli matrices defined as $X=\begin{pmatrix}0&1\\1&0\end{pmatrix}$, $Z=\begin{pmatrix}1&0\\0&-1\end{pmatrix}$. The efficiency of the protocol can be evaluated through teleportation fidelity. The higher the teleportation fidelity is, the more effective the protocol is. Since the initial state is pure, the fidelity could be written as

$$f={}_{S_1S_2}\left\langle\psi\right|\rho_{B_1B_2}\left|\psi\right\rangle_{S_1S_2}, \qquad (17)$$

with $\rho_{B_1B_2}$ being the state that Bob obtains in the last step of the teleportation protocol. To obtain the average fidelity of all possible input states, we set $\alpha_{00}=\cos\eta_3$, $\alpha_{01}=\sin\eta_3\cos\eta_2$, $\alpha_{10}=\sin\eta_3\sin\eta_2\cos\eta_1$, and $\alpha_{11}=\sin\eta_3\sin\eta_2\sin\eta_1$. According to [15], the formula calculating average fidelity takes the following form

$$\left\langle F\right\rangle=\frac{3!}{\pi^3}\prod_{k=1}^{3}\int_0^{\pi/2}\cos\eta_k\left(\sin\eta_k\right)^{2k-1}d\eta_k\prod_{j=1}^{3}\int_0^{2\pi}d\varphi_j f, \qquad (18)$$

where $\eta_k\in[0,\pi/2]$ and $\varphi_j\in[0,2\pi)$.

In the absence of noise, the average fidelity of the protocol is given by

$$\left\langle F\right\rangle=\frac{1}{5}+\frac{1}{5}\left(1+\sin2\theta_1\right)\left(1+\sin2\theta_2\right)=\frac{1}{5}+\frac{1}{5}\left(1+C_{10}\right)\left(1+C_{20}\right), \qquad (19)$$

where $C_{10}=\sin(2\theta_1)$, $C_{20}=\sin(2\theta_2)$ are the concurrences [14] corresponding to the entanglement measures of the parallel states $\left|Q(\theta_1)\right\rangle_{A_1B_1}$; $\left|Q(\theta_2)\right\rangle_{A_2B_2}$. Thus, the average fidelity given by Eq. (19) depends solely on the entanglement of the parallel states $\left|Q(\theta_1)\right\rangle_{A_1B_1}$; $\left|Q(\theta_2)\right\rangle_{A_2B_2}$ that form the quantum channel for the teleportation protocol. In the case where the quantum channel shared between the sender and receiver is maximally entangled ($\theta_1 = \theta_2 = \pi/4$), Bob perfectly recovers the state (1), representing the arbitrary two-qubit state S_1 and S_2. Consequently, both the fidelity and the success probability of the protocol are equal to 1, and the teleportation protocol is considered ideal or perfect. Conversely, when the parallel states $\left|Q(\theta_1)\right\rangle_{A_1B_1}$; $\left|Q(\theta_2)\right\rangle_{A_2B_2}$ are separable (non-entangled), the average fidelity (17) reduces to the best fidelity achievable by classical means, denoted as $\langle F\rangle_c = 2/5$. Furthermore, if we choose $\theta_1 = \theta_2 = \theta$, then $C_{10} = C_{20} = C_0$. The average fidelity (17) simplifies to

$$\langle F\rangle = \frac{2}{5} + \frac{2}{5}C_0 + \frac{1}{5}C_0^2. \tag{20}$$

In the case where the qubits of the quantum channel held by the sender and receiver are subjected to amplitude-damping noise, the shared quantum channel between the two parties takes the form

$$\rho_{A_1B_1A_2B_2}(\theta_1,\theta_2,p_1,p_2) = \rho_{A_1B_1}(\theta_1,p_1,p_2) \otimes \rho_{A_2B_2}(\theta_2,p_1,p_2), \tag{21}$$

where the two parallel states $\rho_{A_1B_2}(\theta_1,p_1,p_2)$ and $\rho_{A_2B_2}(\theta_2,p_1,p_2)$ are explicitly given by the following expressions, respectively

$$\begin{aligned}\rho_{A_1B_1}(\theta_1,p_1,p_2) &= \left(\cos^2\theta_1 + p_1p_2\sin^2\theta_1\right)|00\rangle_{AB}\langle 00| + \sin\theta_1\cos\theta_1\sqrt{(1-p_1)(1-p_2)} \\ &\left(|00\rangle_{AB}\langle 11| + |11\rangle_{AB}\langle 00|\right) + p_1(1-p_2)\sin^2\theta_1|01\rangle_{AB}\langle 01| + (1-p_1)p_2\sin^2\theta_1|10\rangle_{AB}\langle 10| \\ &+ (1-p_1)(1-p_2)\sin^2\theta_1|11\rangle_{AB}\langle 11|.\end{aligned} \tag{22}$$

and

$$\begin{aligned}\rho_{A_2B_2}(\theta_2,p_1,p_2) &= \left(\cos^2\theta_2 + p_1p_2\sin^2\theta_2\right)|00\rangle_{AB}\langle 00| + \sin\theta_2\cos\theta_2\sqrt{(1-p_1)(1-p_2)} \\ &\left(|00\rangle_{AB}\langle 11| + |11\rangle_{AB}\langle 00|\right) + p_1(1-p_2)\sin^2\theta_2|01\rangle_{AB}\langle 01| + (1-p_1)p_2\sin^2\theta_2|10\rangle_{AB}\langle 10| \\ &+ (1-p_1)(1-p_2)\sin^2\theta_2|11\rangle_{AB}\langle 11|.\end{aligned} \tag{23}$$

In this case, the average fidelity of the protocol can be explicitly expressed as

$$\begin{aligned}\langle F\rangle = \frac{1}{5} + \frac{1}{5}&\left[1 + (2p_1p_2 - p_1 - p_2)\sin^2\theta_1 + \sqrt{(1-p_1)(1-p_2)}\sin 2\theta_1\right] \\ &\left[1 + (2p_1p_2 - p_1 - p_2)\sin^2\theta_2 + \sqrt{(1-p_1)(1-p_2)}\sin 2\theta_2\right].\end{aligned} \tag{24}$$

By substituting the expressions for concurrence (13), purity (10), and the multipartite entanglement measure 4-tangle (8) of each state in the pair of parallel states forming the quantum channel (21) into Eq. (24), we obtain the dependence of the average quantum fidelity of the protocol on these characteristic quantities. This provides a comprehensive analysis within the arbitrary two-qubit state teleportation framework through a quantum channel whose qubits are subjected to amplitude-damping noise.

$$\langle F\rangle = \frac{1}{5}+\frac{1}{5}\left(\frac{1}{2}+C_1+\frac{\sqrt{\tau_{41}}+\sqrt{\frac{\tau_{41}}{2}+2P_1-1}}{2}\right)\left(\frac{1}{2}+C_2+\frac{\sqrt{\tau_{42}}+\sqrt{\frac{\tau_{42}}{2}+2P_2-1}}{2}\right), \quad (25)$$

where $C_1=\sqrt{(1-p_1)(1-p_2)}\sin(2\theta_1)-2\sqrt{p_1(1-p_1)p_2(1-p_2)}\sin^2\theta_1,$ (26)

$$C_2=\sqrt{(1-p_1)(1-p_2)}\sin(2\theta_2)-2\sqrt{p_1(1-p_1)p_2(1-p_2)}\sin^2\theta_2, \quad (27)$$

are the concurrence measuring the entanglement of the corresponding states (22) and (23).

$$\begin{aligned}P_1 &= 1-2(p_1+p_2-2p_1p_2)\cos^2\theta_1\sin^2\theta_1\\ &\quad -2\left[p_1(1-p_1)+p_2(1-p_2)-2p_1(1-p_1)p_2(1-p_2)\right]\sin^4\theta_1,\end{aligned} \quad (28)$$

$$\begin{aligned}P_2 &= 1-2(p_1+p_2-2p_1p_2)\cos^2\theta_2\sin^2\theta_2\\ &\quad -2\left[p_1(1-p_1)+p_2(1-p_2)-2p_1(1-p_1)p_2(1-p_2)\right]\sin^4\theta_2.\end{aligned} \quad (29)$$

The purity corresponds to states (22) and (23).

$$\tau_{41}=16p_1(1-p_2)(1-p_1)p_2\sin^4\theta_1, \quad (30)$$

$$\tau_{42}=16p_1(1-p_2)(1-p_1)p_2\sin^4\theta_2. \quad (31)$$

These quantities correspond to the four-qubit multipartite entanglement present in the following quantum states

$$\begin{aligned}\left|Q_1(\theta,p_1,p_2)\right\rangle_{A_1E_{11}B_1E_{21}} &= \cos\theta_1|0000\rangle_{A_1E_{11}B_1E_{21}}+\sin\theta_1\Big[\sqrt{(1-p_1)(1-p_2)}|1010\rangle\\ &\quad +\sqrt{(1-p_1)p_2}|1001\rangle+\sqrt{p_1(1-p_2)}|0110\rangle+\sqrt{p_1p_2}|0101\rangle\Big]_{A_1E_{11}B_1E_{21}},\end{aligned} \quad (32)$$

and

$$\begin{aligned}\left|Q_2(\theta,p_1,p_2)\right\rangle_{A_2E_{12}B_2E_{22}} &= \cos\theta_2|0000\rangle_{AE_1BE_2}+\sin\theta_2\Big[\sqrt{(1-p_1)(1-p_2)}|1010\rangle\\ &\quad +\sqrt{(1-p_1)p_2}|1001\rangle+\sqrt{p_1(1-p_2)}|0110\rangle+\sqrt{p_1p_2}|0101\rangle\Big]_{A_2E_{12}B_2E_{22}}.\end{aligned} \quad (33)$$

From Eq. (25), it can be seen that the average quantum fidelity of the teleportation protocol for an arbitrary two-qubit state via a quantum channel constructed from the pair of parallel states (21), whose initial state is the Bell-like state (2), depends not only on the entanglement and purity of each parallel state in the pair but also on the multipartite entanglement generated between the qubits of the initial Bell-like states and their corresponding environmental qubits.

To optimize the average quantum fidelity expressed in Eq. (25) and to elucidate the physical mechanism responsible for such optimization, we consider two specific scenarios.

2.2.2. First scenario

The amplitude-damping noise acts identically on each qubit, i.e., the noise strengths for both local environments are equal ($p_1 = p_2 = p$). In this case,

$$\tau_{41} = 16p^2(1-p)^2\sin^4\theta_1, \tag{34}$$

$$\tau_{42} = 16p^2(1-p)^2\sin^4\theta_2, \tag{35}$$

$$P_1 = \left[1-2p(1-p)\sin^2\theta_1\right]^2, \tag{36}$$

$$P_2 = \left[1-2p(1-p)\sin^2\theta_2\right]^2. \tag{37}$$

Substituting the expressions from Eqs. (34) to (37) into Eq. (25), the average quantum fidelity in this scenario takes the form:

$$\langle F\rangle = \frac{2}{5}+\frac{1}{5}(C_1+C_2)+\frac{1}{5}C_1C_2, \tag{38}$$

where $$C_1 = (1-p)\sin(2\theta_1)-2p(1-p)\sin^2\theta_1, \tag{39}$$

and $$C_2 = (1-p)\sin(2\theta_2)-2p(1-p)\sin^2\theta_2. \tag{40}$$

In this case, the average fidelity of the protocol depends only on the entanglement of the parallel states that form the quantum channel shared between the sender and the receiver. If the two initial Bell-like states are chosen to be identical ($\theta_1 = \theta_2 = \theta$), then:

$$\langle F\rangle = \frac{2}{5}+\frac{2}{5}C+\frac{1}{5}C^2, \tag{41}$$

where $$C = (1-p)\sin(2\theta)-2p(1-p)\sin^2\theta. \tag{42}$$

In this case, the average fidelity reaches its maximum value when the entanglement of each state in the pair of parallel states forming the quantum channel is maximized. If each qubit of the channel is subject to a specific amplitude-damping noise, i.e., the noise parameter p is fixed, then it is necessary to determine the value θ of the initial parameter of the Bell-like states, such that the average fidelity (41) is optimized. The entanglement (42) and the average fidelity (41) attain their maximal values, referred to as optimal values, when the initial parameter of the channel is chosen to depend on the noise intensity as follows:

$$\theta_{opt} = \frac{1}{2}\arctan\frac{1}{p}, \tag{43}$$

In this case, the entanglement of each state in the pair of parallel channel states is given by

$$C_{opt} = (1-p)\left(\sqrt{1+p^2}-p\right). \tag{44}$$

The fidelity, in this case, is given by

$$\langle F\rangle_{opt} = \frac{2}{5}+\frac{1}{5}(1-p)\left(\sqrt{1+p^2}-p\right)\left[2+(1-p)\left(\sqrt{1+p^2}-p\right)\right]. \tag{45}$$

To compare the optimal average fidelity (45) with the average fidelity in the case where the initial channel consists of maximally entangled Bell states [7], i.e. $\theta_1=\theta_2=\pi/4$, and the noisy environments are symmetric ($\mathrm{p}_1=\mathrm{p}_2=\mathrm{p}$), the average fidelity of the protocol in that case, is given by

$$\langle F\rangle_{0\max} = \frac{2}{5}+\frac{1}{5}\left(2-2p+p^2\right)^2. \tag{46}$$

In this case, the average quantum fidelity obtained from Eq. (46) matches the result reported in [7]. The entanglement of each state in the pair of parallel channel states is given by

$$C_{0\max} = (1-p)^2. \tag{47}$$

To compare the average fidelities given by equations (45) and (46), we plot the results in Figure 1. When the free parameter of the initial states in the pair of parallel states forming the quantum channel is chosen appropriately according to the noise intensity, the average fidelity of the teleportation protocol consistently exceeds that obtained when the initial states of the parallel channel are maximally entangled Bell states [7], over the entire range of noise intensities considered.

2.2.3. Second scenario

In the scenario where the transmission distances of the qubits in the quantum channel or the duration each qubit is subjected to amplitude damping noise are completely determined, meaning the noise intensities p_1 and p_2 are known and distinct, the optimal average quantum fidelity is obtained by following the method outlined in [16]. Specifically, the free parameter θ of the initial states is optimized such that the average quantum fidelity given in equation (24) is maximized. The condition for the average fidelity in (24) to reach its maximum requires the first derivative of the average fidelity with respect to θ_k vanish, i.e., $\partial\langle F\rangle/\partial\theta_k\big|_{\theta_k=(\theta_k)_{opt}}=0,\ k=1,2$, and the second derivative with respect to θ be negative at $\theta_k=(\theta_k)_{opt}$, namely, $\partial^2\langle F\rangle/\partial\theta^2\big|_{\theta_k=(\theta_k)_{opt}}<0$. From these conditions, we determine the optimal value of the free parameter θ_k for the pair of parallel Bell-like states forming the initial quantum channel that maximizes the average quantum fidelity (24), corresponding to the differing noise intensities of the two independent environments. We refer to this value as the optimal free parameter of the initial parallel channel states relative to the noise intensities of the two environments.

$$(\theta_1)_{\mathrm{opt}} = (\theta_2)_{\mathrm{opt}} = \frac{1}{2}\arctan\left(\frac{2\sqrt{(1-\mathrm{p}_1)(1-\mathrm{p}_2)}}{\mathrm{p}_1+\mathrm{p}_2-2\mathrm{p}_1\mathrm{p}_2}\right). \tag{48}$$

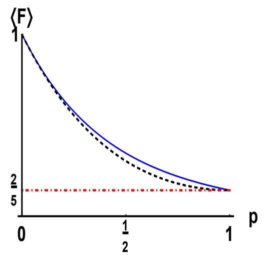

Figure 1. Dependence of the average fidelity of the two-qubit arbitrary state teleportation protocol through identical amplitude-damping noisy environments as a function of noise intensity. The solid blue line represents the optimal average fidelity (45), the dashed line corresponds to the average fidelity with the initial channel states as maximally entangled Bell states (46), and the red dash-dot line indicates the classical fidelity

The optimal average quantum fidelity corresponding to two amplitude-damping noisy environments with different noise intensities, and evaluated at the optimal value of the free parameter of the initial state (14) of the channel (21), which satisfies Eq. (48), is expressed as follows

$$\langle F\rangle_{2\mathrm{opt}} = \frac{1}{5} + \frac{1}{20}\left[2 - p_1 - p_2 + 2p_1p_2 + \sqrt{4(1-p_1)(1-p_2) + (p_1 + p_2 - 2p_1p_2)^2}\right]. \quad (49)$$

In contrast, if the initial state (14) of the quantum channel (21) is chosen to be a maximally entangled Bell state ($\theta_1 = \theta_2 = \theta_{0\max} = \pi/4$), [41] then the corresponding average quantum fidelity (24) takes the form:

$$\langle F\rangle_{20\max} = \frac{1}{5} + \frac{1}{5}\left[1 + \frac{1}{2}(2p_1p_2 - p_1 - p_2) + \sqrt{(1-p_1)(1-p_2)}\right]^2, \quad (50)$$

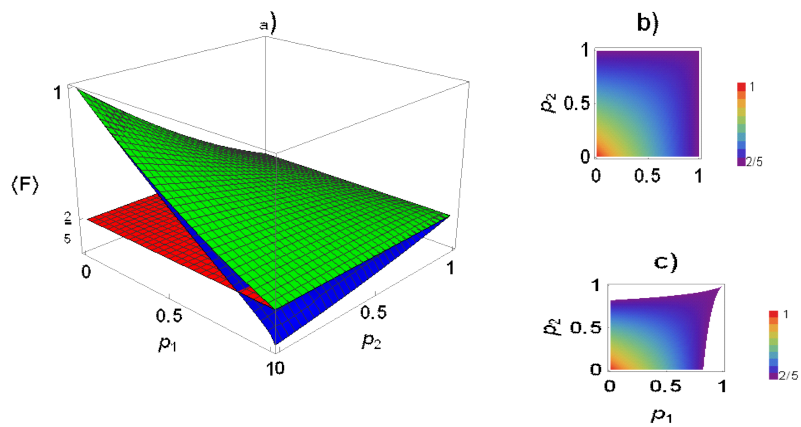

Figure 2. Dependence of the average fidelity on the noise strengths and corresponding phase diagrams. a) The green surface represents the optimal average fidelity (49), while the blue surface corresponds to the average fidelity when the initial states of the channel are maximally entangled Bell states (50). b)** and **c) show the respective phase diagrams for the fidelities given by (49) and (50)

To compare the optimal average quantum fidelity (49) with the average fidelity (50), we plot their values and corresponding phase diagrams as functions of the amplitude-damping noise strengths in Figure 2. From Figure 2(a), it is evident that when the free parameters of the initial states forming the parallel quantum channel are chosen in accordance with the noise strengths, the optimal average fidelity of the protocol always exceeds that obtained when the initial states are maximally entangled Bell states, across the entire range of noise strengths. Additionally, this optimal fidelity is always greater than or at least equal to the best achievable classical fidelity in the whole domain of noise strengths acting on each qubit of the channel, as shown in Figure 2(b). On the contrary, when the initial states forming the parallel quantum channel are chosen as maximally entangled Bell states, the corresponding average fidelity of the protocol falls below the classical fidelity threshold in certain regions. These regions expand significantly when the noise strength affecting one qubit is much larger than that affecting the other, revealing the asymmetry of the noisy environment due to unequal damping strengths. This behavior is clearly reflected in Eqs. (25) and (38), which show that when the noise strengths are equal, the average quantum fidelity depends solely on the entanglement of the pair of parallel states forming the channel. However, when the noise strengths differ, the fidelity depends not only on the entanglement but also on the purity of these two states and the multipartite entanglement generated between the channel qubits and the environmental qubits due to their interactions.

To evaluate which quantity among those the average quantum fidelity (25) depends on contributes most significantly to the enhancement of fidelity-specifically, the purity and entanglement measures of each component state forming the quantum channel and the multipartite entanglement generated between the qubits of each component state and their respective environments - we plotted the differences in these quantities for two values of the free parameter in the initial states forming the parallel channel: the optimal value obtained in Eq. (48) and the value $\theta_1 = \theta_2 = \theta_{0\max} = \pi/4$, as shown in Figure 3.

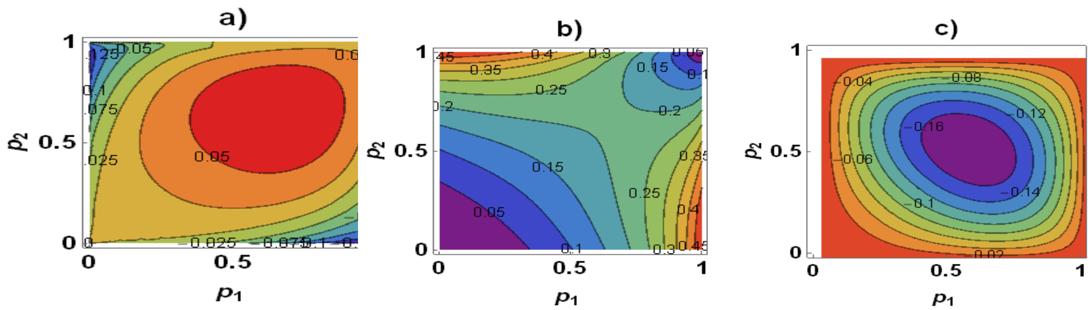

Figure 3. The dependence of the differences in the quantities on which the average quantum fidelity (25) relies, corresponding to two different values of the free parameter of the initial states forming the parallel channel: the optimal value given by Eq. (48), and $\theta_1 = \theta_2 = \theta_{0\max} = \pi/4$. a) Concurrence measure (26), b) Purity measure (28), c) Multipartite entanglement measure (30)

From Figure 3, we observe the following: The difference in the entanglement of each component state forming the channel can be either negative or positive depending on the noise strength (Figure 3a); the difference in purity of each component state is always positive for all values of the noise strength (Figure 3b); in contrast, the difference in multipartite entanglement is always negative over the entire range of noise strengths (Figure 3c). These results indicate that when the free parameter of the initial states forming the parallel channel is appropriately chosen in accordance with the noise strength, the average fidelity of the protocol is consistently higher than that obtained when using maximally entangled Bell states as the initial states, across the entire domain of noise strengths. In this case, the improvement in average fidelity is primarily due to the enhanced purity of each component state forming the quantum channel.

3. Conclusions

In this paper, we investigated the quantum teleportation protocol of an arbitrary two-qubit state through an amplitude-damping noise environment, where the initial states of the pair of parallel states forming the quantum channel are chosen as Bell-like states. Each qubit of the quantum channel is independently affected by amplitude-damping noise. An analytical expression was derived for the average quantum fidelity of the protocol, showing its dependence on the purity and entanglement of each component state in the parallel channel and the multipartite entanglement between the qubits of the channel and their respective environments. Based on these dependencies, we identified the physical mechanisms responsible for enhancing the average fidelity of the protocol under two distinct scenarios:

i) When the noise affecting each qubit of the channel is identical, i.e., the noise strengths are equal ($p_1 = p_2 = p$), the average fidelity depends only on the entanglement of each component state in the parallel channel. In this case, improving the average fidelity is directly linked to increasing the entanglement of the component states. If the free parameter of the initial Bell-like states is chosen appropriately with respect to the noise strength, the resulting average fidelity is always higher than that obtained when the channel consists of maximally entangled Bell states across the entire range of noise strengths.

ii) When the noise affecting the two qubits of the product states in the channel is unequal $(p_1 \neq p_2)$, the average fidelity depends not only on the entanglement of the component states but also on their purity and the multipartite entanglement between the channel qubits and the environmental qubits. In this scenario, enhancing the average fidelity requires simultaneous optimization of all the quantities it depends on. Our analysis shows that by appropriately choosing the free parameter of the initial states to match the asymmetric noise strengths, the average fidelity again surpasses that of the maximally entangled Bell state case throughout the full range of noise values. Moreover, in this case, the fidelity improvement is mainly due to the increased purity of the component states in the parallel channel. Our results shed light on the underlying physical mechanism responsible for improving the average fidelity of quantum teleportation of arbitrary two-qubit states through amplitude-damping noisy environments.

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