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STABILIZATION OF DISCRETE 2D LINEAR SYSTEMS IN FORNASINI-MARCHESINI MODEL

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Abstract. This paper is concerned with the stabilization problem of discrete 2D linear systems described by the second Fornasini-Marchesini model. A necessary and sufficient condition involving the characteristic polynomial is first quoted by which the unforced system is structurally or exponentially stable. On the basis of the derived stability condition, a tractable condition is formulated in the form of linear matrix inequality for obtaining the controller gain of a desired stabilizing state-feedback controller.
Keywords: 2D systems, Fornasini-Marchesini model, state-feedback controller, stabilization.

1. Introduction

Consider the following control system

$$x(i+1,j+1) = A_1x(i+1,j) + A_2x(i,j+1) + Bu(i,j), \tag{1.1}$$

where $x(i,j) \in \mathbb{R}^n$ is the state vector, $u(i,j) \in \mathbb{R}^m$ is the control input vector and A_1, A_2, B are real matrices of appropriate dimensions. In system (1.1), the dynamic propagation is specified in two independent directions in two time scales defined by i and j. By which, system (1.1) belongs to a special dynamical systems called two-dimensional (2D) systems [1]. In addition, the 2D linear system (1.1) is described by the so-called Fornasini-Marchesini model (FM model). In the past few decades, 2D systems have drawn wide attention due to their extensive applications in many fields such as signal filtering, image processing, or repetitive processes. Many studies of 2D systems have been reported in the literature, especially the stability analysis of 2D systems. We refer the reader to recent works [1]-[5] and references therein.

For given initial sequences $x(0, j)$ and $x(i, 0)$, the unique state trajectory of system (1.1) can be represented by

$$x(i+1, j+1) = \sum_{k=0}^{i} \Phi(k, j) A_1 x(i+1-k, 0) + \sum_{k=0}^{j} \Phi(i, k) A_2 x(0, j+1-k), \quad (1.2)$$

where the fundamental matrix $\Phi(i, j)$ is defined inductively by

$$\Phi(i+1, j+1) = \Phi(i, j+1) A_2 + \Phi(i+1, j) A_1,$$
$$\Phi(i, 0) = A_2, \quad \Phi(0, j) = A_1.$$

The unforced system of (1.1) is said to be structurally stable (FM-SS) if

$$\det(\lambda\mu I_n - \lambda A_1 - \mu A_2) \neq 0$$

for all $\lambda, \mu \in \mathbb{C}$ that satisfy $|\lambda| \geq 1$ and $|\mu| \geq 1$. According to [2], [5], system (1.1) is said to be exponentially stable (FM-ES1) if there exist scalars $M > 0$ and $q \in (0, 1)$ such that for any initial sequences $(x(0, j), x(i, 0))$, $(i, j) \in \mathbb{N}^2$, we have

$$\|x(i, j)\| \leq M \left(\sum_{k=0}^{l} \frac{\|x(0, k)\|}{q^{k+1}} + \sum_{k=0}^{i} \frac{\|x(k, 0)\|}{q^{k+1}} \right) q^{i+j}.$$

On the other hand, system (1.1) is exponentially stable in the second sense (FM-ES2) if for any exponentially convergent initial sequences $x(0, j)$, $x(i, 0)$, that is, for all $(i, j) \in \mathbb{N}^2$,

$$\|x(0, j)\| \leq M q^j, \quad \|x(i, 0)\| \leq M q^i$$

with scalars $M > 0$ and $q \in (0, 1)$, there exist a $\tilde{q} \in (0, 1)$ and a $\tilde{M} > 0$ such that

$$\|x(i, j)\| \leq \tilde{M} \tilde{q}^{i+j}.$$

It was shown in [3] that the above stability concepts are indeed equivalent

$$\text{FM-SS} \iff \text{FM-ES1} \iff \text{FM-ES2}.$$

Moreover, system (1.1) is FM-SS if and only if

$$\det(I_n - \lambda A_1 - \mu A_2) \neq 0 \quad (1.3)$$

for all $\lambda, \mu \in \mathbb{C}$ with $|\lambda| \leq 1$ and $|\mu| \leq 1$.

Condition (1.3) provides a characterization for checking stability of open-loop system of (1.1). Assume that the open system of (1.1) is not structurally stable. We aim to design a state-feedback controller (SFC) of the form

$$u(i, j) = K x(i, j), \quad (1.4)$$

where $K \in \mathbb{R}^{n\times m}$ is the controller gain which will be determined. By integrating the controller (1.4), the resulting closed-loop system of (1.1) can be represented as

$$x(i+1,j+1) = A_c x(i,j) + A_1 x(i+1,j) + A_2 x(i,j+1), \tag{1.5}$$

with $A_c = BK$. System (1.5) is typically referred to general FM model by which condition (1.3) cannot be directly adapted. In addition, for the design problem of K, the obtained stability condition through characteristic polynomial is not tractable due to the existence of an unknown matrix K. This motivates us for the present study to deal with the stabilization problem of (1.1).

2. Stability of 2D linear Roesser systems

Consider the following 2D system described by the Roesser model

$$\begin{bmatrix} x^h(i+1,j) \\ x^v(i,j+1) \end{bmatrix} = \underbrace{\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}}_{A} \begin{bmatrix} x^h(i,j) \\ x^v(i,j) \end{bmatrix}, \tag{2.1}$$

where $x^h(i,j) \in \mathbb{R}^{n_1}$, $x^v(i,j) \in \mathbb{R}^{n_2}$ are the horizontal and vertical vector states, respectively, A is a given real matrix of appropriate dimension. Initial conditions of system (2.1) are determined by sequences $x^h(0,j)$ and $x^v(i,0)$ for all $i,j \in \mathbb{N}$. We also denote by

$$x(i,j) = [x^{h\top}(i,j), x^{v\top}(i,j)]^\top \in \mathbb{R}^n, n = n_1 + n_2.$$

For the Roesser system, we have the following important concept.

Definition 2.1. *The 2D Roesser system* (2.1) *is said to be structurally stable (R-SS) if*

$$\det\left(\begin{bmatrix} \lambda I_{n_1} & 0 \\ 0 & \mu I_{n_2} \end{bmatrix} - A\right) \neq 0 \tag{2.2}$$

for all $\lambda, \mu \in \mathbb{C}$ *with* $|\lambda| \geq 1$ *and* $|\mu| \geq 1$.

Definition 2.2. *System* (2.1) *is exponentially stable (R-ES1) if there exists scalars* $M > 0$ *and* $q \in (0,1)$ *such that for any initial sequences* $(x^h(0,j), x^v(i,0))$*, it holds that*

$$\|x(i,j)\| \leq M \left(\sum_{k=0}^{l} \frac{\|x^h(0,k)\|}{q^{k+1}} + \sum_{k=0}^{i} \frac{\|x^v(k,0)\|}{q^{k+1}} \right) q^{i+j}.$$

Definition 2.3. *System* (2.1) *is exponentially stable (R-ES2) if for exponential decaying initial sequences* $x^h(0,j)$*,* $x^v(i,0)$*, that is, for all* $(i,j) \in \mathbb{N}^2$*,*

$$\|x^h(0,j)\| \leq Mq^j, \quad \|x^v(i,0)\| \leq Mq^i$$

with $M > 0$ *and* $q \in (0,1)$*, then there exist* $\tilde{q} \in (0,1)$ *and* $\tilde{M} > 0$ *such that*

$$\|x(i,j)\| \leq \tilde{M}\tilde{q}^{i+j}.$$

It was shown using the connection between various stability concepts for the Roesser model and FM model that, for 2D Roesser system (2.1), the concepts of R-SS, R-ES1 and R-ES2 are equivalent [3]. Moreover, we have the following result which is essential for our design problem in the next section.

Theorem 2.1. *The Roesser system* (2.1) *is structurally stable if and only if*

$$\det\left(I_{n_1+n_2} - \begin{bmatrix} \lambda I_{n_1} & 0 \\ 0 & \mu I_{n_2} \end{bmatrix} A\right) \neq 0 \tag{2.3}$$

for all $\lambda, \mu \in \mathbb{C}$ *satisfying* $|\lambda| \leq 1$, $|\mu| \leq 1$.

Corollary 2.1. *If the Roesser system* (2.1) *is structurally stable, then the moduli of the eigenvalues of matrices* A_{11} *and* A_{22} *must lie within the unit disk of the complex plane* $\mathbb{C}$. *In other words, the matrices* A_{11} *and* A_{22} *are Schur stable (spectral radius less than 1).*

3. Controller design

In this section, we address the design problem of a desired SFC (1.4) by which the closed-loop system (1.5) is structurally stable (or equivalent FM-ES1 and FM-ES2). Consider the closed-loop system (1.5). By using the following state transformations

$$\begin{aligned} x^h(i,j) &= x(i+1,j) - A_1 x(i,j), \\ x^v(i,j) &= x(i,j), \end{aligned} \tag{3.1}$$

system (1.5) can be represented by the Roesser model

$$\begin{bmatrix} x^h(i+1,j) \\ x^v(i,j+1) \end{bmatrix} = \underbrace{\begin{bmatrix} A_2 & A_c + A_1 A_2 \\ I_n & A_1 \end{bmatrix}}_{\mathcal{A}} \begin{bmatrix} x^h(i,j) \\ x^v(i,j) \end{bmatrix}. \tag{3.2}$$

According to the transformations in (3.1), it can be verified that (1.5) is structurally stable (corresponding to FM-ES1 and FM-ES2) if and only if the Roesser system (3.2) is structurally stable (corresponding to R-ES1 and R-ES2 stable). Based on the Roesser system model (3.2), we find tractable conditions for the design of a stabilizing controller (1.4).

Let us denote the block matrix $\mathcal{A}$ by

$$\mathcal{A} = \begin{bmatrix} A_2 & A_c + A_1 A_2 \\ I_n & A_1 \end{bmatrix} \triangleq \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}.$$

According to Theorem 2.1, the Roesser system (3.2) is structurally stable (corresponding to R-ES1 and R-ES2 stable) if and only if

$$\det \begin{bmatrix} I_n - z_1 A_{11} & -z_1 A_{12} \\ -z_2 A_{21} & I_n - z_2 A_{22} \end{bmatrix} \neq 0 \tag{3.3}$$

for all $z_1, z_2 \in \mathbb{C}$ with $|z_1| \leq 1$ and $|z_2| \leq 1$.

The following result is a special connection from characteristic polynomial condition to LMI setting.

Theorem 3.1. *Condition* (3.3) *holds if there exist symmetric positive definite matrices* P, Q *such that*

$$\mathcal{A}^\top \mathcal{D}_{P,Q} \mathcal{A} - \mathcal{D}_{P,Q} < 0, \tag{3.4}$$

where $\mathcal{D}_{P,Q} = \mathrm{diag}(P, Q)$.

Proof. In contrast, assume that there exists a point $z = (z_1, z_2)$ in the unit disk $\mathbb{D}$ defined by

$$\mathbb{D} = \{(z_1, z_2) : z_1, z_2 \in \mathbb{C}, |z_1| \leq 1, |z_2| \leq 1\}$$

such that

$$\begin{vmatrix} I_n - z_1 A_{11} & -z_1 A_{12} \\ -z_2 A_{21} & I_n - z_2 A_{22} \end{vmatrix} = 0.$$

Then, there is a vector $\mathcal{X} \in \mathbb{R}^{2n}$ such that

$$\begin{bmatrix} I_n - z_1 A_{11} & -z_1 A_{12} \\ -z_2 A_{21} & I_n - z_2 A_{22} \end{bmatrix} \mathcal{X} = 0$$

and hence

$$\mathcal{X} = \mathbb{I}(z_1, z_2) \mathcal{A} \mathcal{X},$$

where $\mathbb{I}(z_1, z_2) = \mathrm{diag}(z_1 I_n, z_2 I_n)$. Let $\mathcal{S} = \mathcal{D}_{P,Q} - \mathcal{A}^\top \mathcal{D}_{P,Q} \mathcal{A} > 0$ and denote by $\mathcal{X}^*$, z^* the conjugate of $\mathcal{X}$ and $z = (z_1, z_2)$. we have,

$$\begin{aligned} \mathcal{X}^* \mathcal{D}_{P,Q} \mathcal{X} &= \mathcal{X}^* \mathcal{A}^\top \mathbb{I}(z_1^*, z_2^*) \mathcal{D}_{P,Q} \mathbb{I}(z_1, z_2) \mathcal{A} \mathcal{X} \\ &= \mathcal{X}^* \mathcal{A}^\top \mathbb{I}(|z_1|^2, |z_2|^2) \mathcal{D}_{P,Q} \mathcal{A} \mathcal{X} \\ &= \mathcal{X}^* \mathcal{D}_{P,Q} \mathcal{X} - \mathcal{X}^* \mathcal{S} \mathcal{X} \\ &\quad - \mathcal{X}^* \mathcal{A}^\top \mathbb{I}(1 - |z_1|^2, 1 - |z_2|^2) \mathcal{D}_{P,Q} \mathcal{A} \mathcal{X} \\ &\leq \mathcal{X}^* \mathcal{D}_{P,Q} \mathcal{X} - \mathcal{X}^* \mathcal{S} \mathcal{X}. \end{aligned}$$

This yields $\mathcal{X} = 0$ due to $\mathcal{S} > 0$. This contradicts with $\mathcal{X} \neq 0$. □

Condition (3.4) is quadratic with respect to the control parameter matrix K. Thus, it is not feasible for the design problem. To find K, we define the matrix variables

$$\widetilde{P} = P^{-1}, \ \widetilde{Q} = Q^{-1}$$

and $\mathcal{D}_{\widetilde{P},\widetilde{Q}} = \mathrm{diag}(\widetilde{P}, \widetilde{Q})$. Note that,

$$\mathrm{diag}(\widetilde{P}, \widetilde{Q}) = \mathcal{D}_{P,Q}^{-1} \text{ and } \mathcal{D}_{\widetilde{P},\widetilde{Q}} = \mathcal{D}_{P,Q}^{-1}.$$

By pre- and post-multiplying with $\mathcal{D}_{\widetilde{P},\widetilde{Q}}$, condition (3.4) holds if and only if

$$\mathcal{D}_{\widetilde{P},\widetilde{Q}}\mathcal{A}^\top\mathcal{D}^{-1}_{\widetilde{P},\widetilde{Q}}\mathcal{A}\mathcal{D}_{\widetilde{P},\widetilde{Q}} - \mathcal{D}_{\widetilde{P},\widetilde{Q}} < 0. \tag{3.5}$$

Note further that matrix $\mathcal{A}$ can be written as

$$\mathcal{A} = \underbrace{\begin{bmatrix} A_2 & A_1A_2 \\ I_n & A_1 \end{bmatrix}}_{\hat{A}} + \underbrace{\begin{bmatrix} B \\ 0 \end{bmatrix}}_{\hat{B}} \underbrace{\begin{bmatrix} 0 & K \end{bmatrix}}_{\hat{K}}.$$

By applying the Schur complement lemma, condition (3.5) is equivalent to

$$\begin{bmatrix} -\mathcal{D}_{\widetilde{P},\widetilde{Q}} & \mathcal{D}_{\widetilde{P},\widetilde{Q}}\hat{A}^\top + \mathcal{D}_{\widetilde{P},\widetilde{Q}}\hat{K}^\top\hat{B}^\top \\ \hat{A}\mathcal{D}_{\widetilde{P},\widetilde{Q}} + \hat{B}\hat{K}\mathcal{D}_{\widetilde{P},\widetilde{Q}} & -\mathcal{D}_{\widetilde{P},\widetilde{Q}} \end{bmatrix} < 0. \tag{3.6}$$

Now, by the change of variable

$$\mathcal{D}_{\widetilde{P},\widetilde{Q}}\hat{K}^\top = \mathcal{Z}^\top, \tag{3.7}$$

condition (3.6) becomes

$$\begin{bmatrix} -\mathcal{D}_{\widetilde{P},\widetilde{Q}} & \mathcal{D}_{\widetilde{P},\widetilde{Q}}\hat{A}^\top + \mathcal{Z}^\top\hat{B}^\top \\ \hat{A}\mathcal{D}_{\widetilde{P},\widetilde{Q}} + \hat{B}\mathcal{Z} & -\mathcal{D}_{\widetilde{P},\widetilde{Q}} \end{bmatrix} < 0. \tag{3.8}$$

Due to the structure of the gain matrix K and the variable change (3.7), matrix $\mathcal{Z}$ is designed as

$$\mathcal{Z} = \begin{bmatrix} 0 & \hat{\mathcal{Z}} \end{bmatrix}.$$

Then, from (3.8) we get

$$K = \hat{\mathcal{Z}}\widetilde{Q}^{-1}.$$

These results are summarized in the following theorem.

Theorem 3.2. *The 2D FM system* (1.1) *is stabilizable (in the sense of structural stability FM-SS or exponential stability FM-ES1, FM-ES2) via an SFC* (1.4) *if there exist symmetric positive definite matrices $\widetilde{P}$, $\widetilde{Q}$ and real matrix $\hat{\mathcal{Z}}$ satisfying the linear matrix inequality* (3.8)*. Moreover, the controller gain matrix is given by*

$$K = \hat{\mathcal{Z}}\widetilde{Q}^{-1}.$$

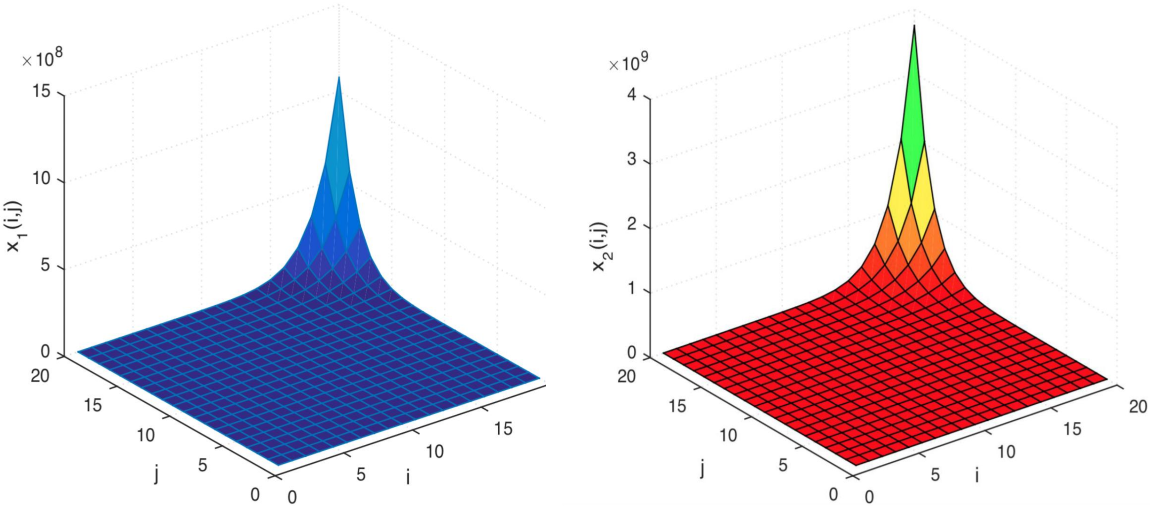
4. Simulations

In this section, we give a numerical example with simulations to illustrate the effectiveness of the design conditions.

Consider a 2D FM system in the form of (1.1) with the system matrices

$$A_1 = \begin{bmatrix} 0.15 & 0.1 \\ 0.66 & 0.75 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0.3 & 0.5 \\ 0.4 & 0.6 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

It can be verified that the open-loop system is unstable. A state trajectory $x^h(i,j) = \begin{bmatrix} x_1^h(i,j) \\ x_2^h(i,j) \end{bmatrix}$ of the open-loop system is presented in Fig. 1(a)-(b). It can be seen that the system state jumps out from the the original surface.



(a) ***(b)***

Figure 1. A state trajectory of the open-loop system

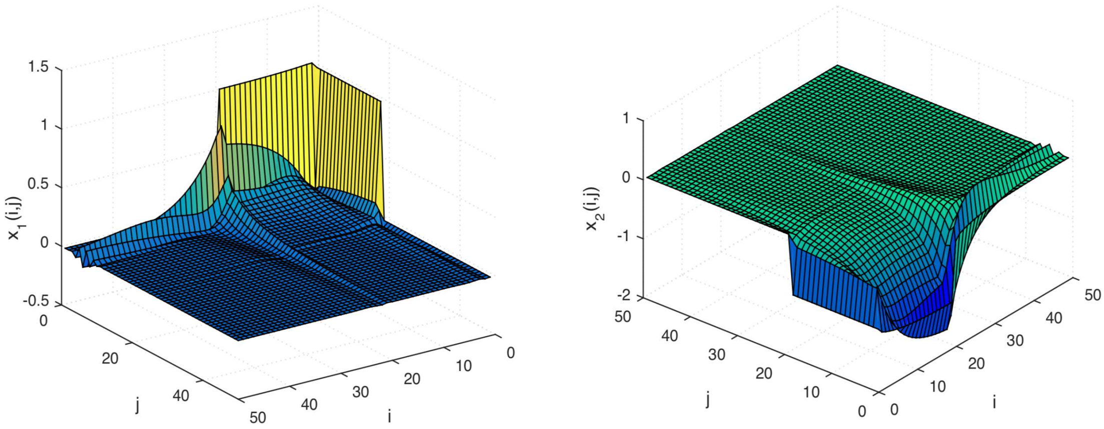
We now utilize the design conditions given in Theorem 3.2. By using the LMI Toolbox to solve (3.8) with respect to matrix variables $\widetilde{P}$, $\widetilde{Q}$ and $\hat{\mathcal{Z}}$ we then obtain

$$\widetilde{P} = \begin{bmatrix} 57.0698 & -36.4908 \\ -36.4908 & 26.3855 \end{bmatrix}, \quad \widetilde{Q} = \begin{bmatrix} 77.8401 & -45.5475 \\ -45.5475 & 84.8703 \end{bmatrix},$$
$$\hat{\mathcal{Z}} = \begin{bmatrix} -2.7858 & -30.2072 \end{bmatrix}.$$

By Theorem 3.2, the controller gain is obtained as

$$K = \hat{\mathcal{Z}} \widetilde{Q}^{-1} = \begin{bmatrix} -0.3558 & -0.5469 \end{bmatrix}.$$

With the obtained controller gain, the closed-loop system (1.5) is stable. A state trajectory of system (1.5) is presented in Fig. 2(a)-(b). It can be seen that the conducted state trajectory converges to the original surface. This demonstrates the effectiveness of the design method.



(a) ***(b)***

Figure 2. A state trajectory of the closed-loop system

5. Conclusions

In this paper, the stabilization problem has been developed for a class of 2D linear systems are described by the second Fornasini-Marchesini model. Based on a necessary and sufficient condition involving the characteristic polynomial, tractable conditions have been formulated in the form of linear matrix inequalities for obtaining the controller gain of a desired stabilizing state-feedback controller. The effectiveness of the derived stabilization conditions has been validated via numerical simulations.

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