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ASYMMETRIC FUNCTIONAL-BASED APPROACH TO EXPONENTIAL STABILITY OF LINEAR DISTRIBUTED TIME-DELAY SYSTEMS

Do Thu Phuong
Faculty of Fundamental Sciences, Hanoi University of Industry, Hanoi city, Vietnam
*Corresponding author: Do Thu Phuong, e-mail: phuongdt@haui.edu.vn

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Abstract. In this paper, the problem of exponential stability of linear time-delay systems with mixed discrete and distributed delays is studied. Based on an asymmetric Lyapunov–Krasovskii functional approach, sufficient conditions are derived in terms of linear matrix inequalities to guarantee the exponential convergence of the system state trajectories with a prescribed decay rate. The efficacy of the obtained results is demonstrated by a given numerical example and simulations.
Keywords: time-delay systems, exponential stability, asymmetric functional.

1. Introduction

Stability theory and its applications in the control theory of time-delay systems is one of the most important research topics [1]. This area has attracted considerable attention during the last decade. A large number of fundamental results on the asymptotic stability of time-delay systems have been established based on variant schemes of Lyapunov–Krasovskii functional (LKF) method and linear matrix inequalities (LMIs) setting [2]. It is recognized that asymptotic stability is a synonym of exponential stability and, in practical applications, it is important to find an estimate of the transient decaying rate of time-delay systems. Therefore, a great deal of efforts has been devoted to studying the exponential stability of time-delay systems (see, for example, [3]-[5]). In particular, the authors of [6] proposed new integral-based inequalities (called weighted integral inequalities) which were later utilizing to derive exponential stability conditions for linear systems with variable delays. The proposed schemes in [6] were also utilized into controller design problems for power multi-area systems with communication delays in [7].

It is noted that the aforementioned results have been derived using symmetric LKFs with positive kernels. This usually produces much conservativeness in the derived stability and design conditions. Recently, an improved approach has been proposed by constructing asymmetric functionals [8]. More specifically, the functional candidate is composed of a quadratic functional and integral terms of which the quadratic term is not necessary to be symmetric and positive definite. The method of [8] was shown to be one that can produce less conservative conditions than existing methods.

In this paper, an asymmetric functional-based method with weighted integral functionals is extended for a class of linear time-delay systems with discrete and distributed delays. Based on the proposed method and some weighted integral inequalities, exponential stability conditions are derived in terms of LMIs which guarantee the exponential convergence of state trajectories of the system with a prescribed decaying rate.

Notation. $\mathbb{R}^n$ and $\mathbb{R}^{m\times n}$ denote the n-dimensional Euclidean space and the set of $m \times n$ real matrices, respectively. A^T is the transpose of a matrix $A \in \mathbb{R}^{n\times m}$, $\mathrm{col}\{u_1, u_2, \ldots, u_k\}$ denotes the augmented vector formulated by stacking components of vectors $u_1, u_2, \ldots, u_k$. A matrix $A \in \mathbb{R}^{n\times n}$ is symmetric if $A = A^T$ and semi-positive definite, write as $A \geq 0$, if it is symmetric and $x^T Ax \geq 0$ for all $x \in \mathbb{R}^n$. If $x^T Ax > 0$ for all $x \in \mathbb{R}^n$, $x \neq 0$, then A is said to be positive definite, write as $A > 0$. Let $\mathbb{S}^n_+ = \{A \in \mathbb{R}^{n\times n} : A > 0\}$. The notation $*$ represents symmetry terms in a symmetric matrix.

2. Preliminaries

Let $C([-h, 0], \mathbb{R}^n)$ be the Banach space of $\mathbb{R}^n$-valued continuous functions defined on the interval $[-h, 0]$ endowed with the norm

$$\|\phi\| = \sup_{-h\leq s\leq 0} \|\phi(s)\|$$

for a function $\phi \in C([-h, 0], \mathbb{R}^n)$. Consider the following functional differential equation

$$\begin{aligned} &\dot{x}(t) = f(t, x_t), \quad t \geq t_0, \\ &x_{t_0} = \phi, \end{aligned} \tag{2.1}$$

where $f : D = [t_0, \infty) \times C([-h, 0], \mathbb{R}^n) \to \mathbb{R}^n$ is a continuous function, $\phi \in C([-h, 0], \mathbb{R}^n)$ is the initial function and x_t denotes the state segment $\{x(t + s) : -h \leq s \leq 0\}$, that is, $x_t \in C([-h, 0], \mathbb{R}^n)$ for each $t \geq t_0$. We assume that the function f satisfies conditions such that for any $\phi \in C([-h, 0], \mathbb{R}^n)$ there exists a unique solution $x(t, \phi)$ of (2.1) which is defined on $[t_0, \infty)$. In addition, we assume $f(t, 0) = 0$ so that system (2.1) has the trivial solution $x = 0$.

Definition 2.1. *The trivial solution* $x = 0$ *of* (2.1) *is said to be globally exponentially stable (GES) if there exist positive constants* α, β *such that any solution* $x(t,\phi)$ *of* (2.1) *satisfies*

$$\|x(t,\phi)\| \leq \beta\|\phi\|e^{-\alpha(t-t_0)}, \quad t \geq t_0.$$

Let $V : \mathbb{R}^+ \times C([-h,0],\mathbb{R}^n) \to \mathbb{R}$ be a continuous function. The derivative of $V(t,\phi)$ along state trajectories of system (2.1) is defined as

$$\dot{V}(t,\phi) = \limsup_{\epsilon \to 0^+} \frac{1}{\epsilon}\left[V(t+\epsilon, x_{t+\epsilon}(t_0,\phi)) - V(t,\phi)\right].$$

Theorem 2.1 (Lyapunov–Krasovskii theorem)**.** *If there exists a functional* $V : \mathbb{R}^+ \times C([-h,0],\mathbb{R}^n) \to \mathbb{R}$ *and positive scalars* λ_1, λ_2, λ_3 *satisfying the following conditions*

(i) $\lambda_1\|x_t(0)\|^2 \leq V(t,x_t) \leq \lambda_2\|x_t\|^2$,

(ii) $\dot{V}(t,x_t) + 2\lambda_3 V(t,x_t) \leq 0$,

where $\|x_t\| = \sup_{-h\leq s\leq 0}\|x(t+s)\|$, *then the trivial solution* $x = 0$ *of* (2.1) *is GES. Moreover, any solution* $x(t,\phi)$ *of* (2.1) *satisfies the following estimate*

$$\|x(t,\phi)\| \leq \sqrt{\frac{\lambda_2}{\lambda_1}}\|\phi\|e^{-\lambda_3(t-t_0)}, \quad t \geq t_0.$$

Consider a class of linear systems with mixed time-delay given by

$$\begin{aligned} &\dot{x}(t) = A_0x(t) + A_1x(t-h) + A_2\int_{t-h}^{t} x(s)ds, \quad t \geq 0, \\ &x(t) = \phi(t), \quad t \in [-h,0], \end{aligned} \tag{2.2}$$

where A_0, A_1, and $A_2 \in \mathbb{R}^{n\times n}$ are given matrices, $h \geq 0$ is a known scalar representing the time-delay and $\phi \in C([-h,0],\mathbb{R}^n)$ is the initial condition. Our main aim here is to derive conditions by which system (2.2) is exponentially stable with a prescribed decay rate. First, to manipulate the derivative of LKF candidates, we introduce the following lemmas.

Lemma 2.1 (Hien and Trinh [6])**.** *For a given matrix* $R \in \mathbb{S}_n^+$, *scalars* $b > a$, $\alpha > 0$ *and a function* $\varphi \in C([a,b],\mathbb{R}^n)$, *the following inequalities hold*

$$\int_a^b e^{\alpha(s-b)}\varphi^T(s)R\varphi(s)ds \geq \frac{\alpha}{\gamma_0}\left(\int_a^b \varphi(s)ds\right)^T R\left(\int_a^b \varphi(s)ds\right), \tag{2.3}$$

$$\int_a^b\int_s^b e^{\alpha(u-b)}\varphi^T(u)R\varphi(u)duds$$

$$\geq \frac{\alpha^2}{\gamma_1}\left(\int_a^b\int_s^b \varphi(u)duds\right)^T R\left(\int_a^b\int_s^b \varphi(u)duds\right), \tag{2.4}$$

where γ_k, $k \geq 0$, denotes the residual $e^{\alpha(b-a)} - \sum_{j=0}^{k}\frac{\alpha^j(b-a)^j}{j!}$.

Lemma 2.2 (Hien and Trinh [6])**.** *For a function $\omega : [a,b] \to \mathbb{R}^n$ with the derivative $\dot{\omega}$ belongs to $C([a,b],\mathbb{R}^n)$ and given matrix $R > 0$, the following inequality holds*

$$\int_a^b e^{2\alpha(s-b)}\dot{\omega}^T(s)R\dot{\omega}(s)ds \geq \frac{1}{\rho(\alpha)}(\omega(b)-\omega(a))^T R(\omega(b)-\omega(a)) + (\frac{\eta_\alpha}{\sqrt{\beta_\alpha}})^2\zeta_\alpha^T R\zeta_\alpha, \tag{2.5}$$

where

$$\zeta_\alpha = \omega(b) + \left(\frac{1}{l\eta_\alpha} - 1\right)\omega(a) - \frac{1}{l\eta_\alpha}\int_a^b \omega(s)ds,$$
$$l = b-a, \quad \eta_\alpha = 1 - \frac{1}{2\alpha l} + \frac{1}{e^{2\alpha l}-1},$$
$$\beta_\alpha = \frac{1}{2\alpha}\left[\frac{e^{2\alpha l}-1}{(2\alpha l)^2} - \frac{1}{e^{2\alpha l}-1} - 1\right],$$
$$\rho(\alpha) = \frac{e^{2\alpha(b-a)}-1}{2\alpha}.$$

In the critical case, when α approaches zero, the inequalities (2.3)-(2.4) are reduced to the following ones.

Lemma 2.3. *For a function $\omega : [a,b] \to \mathbb{R}^n$, scalar $b > a$ and a given matrix $Q > 0$ of appropriate dimensions, the following inequalities hold*

$$\int_a^b \omega^T(s)Q\omega(s)ds \geq \frac{1}{b-a}\left(\int_a^b \omega(s)ds\right)^T R\left(\int_a^b \omega(s)ds\right), \tag{2.6}$$

$$\int_a^b\int_s^b \omega^T(u)Q\omega(u)duds \geq \frac{2}{(b-a)^2}\left(\int_a^b\int_s^b \omega(u)duds\right)^T \times Q\left(\int_a^b\int_s^b \omega(s)ds\right). \tag{2.7}$$

3. Exponential stability conditions: An asymmetric functional approach

For convenience, we define the following notations

$$e_i = [O_{n\times(i-1)} \quad I_n \quad O_{n\times(3-i)n}] \in \mathbb{R}^{n\times 3n}, i = 1,2,3,$$

$$\eta_\alpha = 1 - \frac{1}{2\alpha h} + \frac{1}{e^{2\alpha h} - 1},$$
$$\beta_\alpha = \frac{1}{2\alpha}\left[\frac{e^{2\alpha h} - 1}{(2\alpha h)^2} - \frac{1}{e^{2\alpha h} - 1} - 1\right],$$
$$\rho(\alpha) = \frac{e^{2\alpha h} - 1}{2\alpha}, \quad \gamma = \frac{\eta_\alpha^2}{\beta_\alpha}, \quad \nu = \frac{1}{h\eta_\alpha} - 1.$$

We are now in a position to present the first result of exponential stability of system (2.2) as given in the following theorem.

Theorem 3.1. *For a given $\alpha > 0$, system* (2.2) *is GES with decay rate α if there exist matrices $P = \begin{bmatrix} P_1 & P_2 \end{bmatrix}$ with $P_1 = P_1^T$ and $R \in \mathbb{S}_n^+$ such that*

$$M = \begin{bmatrix} M_{11} & M_{12} & M_{13} \\ * & M_{22} & M_{23} \\ * & * & M_{33} \end{bmatrix} < 0, \tag{3.1}$$

$$D = \begin{bmatrix} P_1 + \frac{1}{\rho(\alpha)}R & \frac{1}{2}P_2 - \frac{1}{\rho(\alpha)}R \\ * & e^{-2\alpha h}Q + \frac{1}{\rho(\alpha)}R \end{bmatrix} > 0, \tag{3.2}$$

where

$$M_{11} = A_0^T P_1 + P_1 A_0 + 2\alpha P_1 + P_2 + Q + hA_0^T R A_0 - \frac{2}{\rho(\alpha)}R - 2\gamma R,$$
$$M_{12} = A_1^T P_1 - \frac{1}{2}P_2 + \frac{1}{2}hA_0^T R A_1 - \frac{1}{\rho(\alpha)}R - \gamma\nu R,$$
$$M_{13} = A_2^T P_1 + \alpha P_2 + \frac{1}{1}A_0 P_2 + \frac{1}{2}hA_0^T R A_2 + \gamma(\nu + 1)R,$$
$$M_{22} = -e^{-2\alpha h}Q + hA_1^T R A_1 - \frac{2}{\rho(\alpha)}R - 2\gamma\nu R,$$
$$M_{23} = \frac{1}{2}A_1^T P_2 + \frac{1}{2}hA_1^T R A_2 + \gamma\nu(\nu + 1)R,$$
$$M_{33} = A_2^T P_2 + hA_2^T R A_2 - 2\gamma(\nu + 1)^2 R.$$

Proof. We consider the following LKF candidate

$$V(x_t) = V_1(t) + V_2(t) + V_3(t), \tag{3.3}$$

where

$$V_1(t) = x^T(t)P\eta(t), \quad \eta(t) = \begin{bmatrix} x^T(t) & \int_{t-h}^{t} x^T(u)du \end{bmatrix}^T,$$
$$V_2(t) = \int_{t-h}^{t} e^{2\alpha(u-t)}x^T(u)Qx(u)du,$$

$$V_3(t) = \int_{t-h}^{t}\int_{s}^{t} e^{2\alpha(u-t)}\dot{x}^T(u)R\dot{x}(u)duds.$$

It can be verified from (3.3) that there exists a positive scalar Λ such that

$$V(x_t) \le \Lambda\|x_t\|^2,\ t \ge 0.$$

Let $\xi(t) = \mathrm{col}\{x(t), x(t-h), \int_{t-h}^{t} x(s)ds\}$. Taking derivative of $V(x_t)$ along trajectories of (2.2), we obtain

$$\begin{aligned}\dot{V}_1(t) + 2\alpha V_1(t) &= \dot{x}^T(t)P\eta(t) + x^T(t)P\dot{\eta}(t) + 2\alpha V_1(t)\\ &= \xi^T(t)M_1\xi(t),\end{aligned} \tag{3.4}$$

and

$$\dot{V}_2(t) = x^T(t)Qx(t) - e^{-2\alpha h}x^T(t-h)Qx(t-h) - 2\alpha V_2(t)$$

and thus

$$\dot{V}_2(t) + 2\alpha V_2(t) = \xi^T(t)M_2\xi(t). \tag{3.5}$$

Next, we have

$$\begin{aligned}\dot{V}_3(t) = &-\int_{t-h}^{t} e^{2\alpha(u-t)}\dot{x}^T(u)R\dot{x}(u)du + \int_{t-h}^{t}\dot{x}^T(t)R\dot{x}(t)ds\\ &-\int_{t-h}^{t} e^{2\alpha(u-t)}\dot{x}^T(u)R\dot{x}(u)du - 2\alpha V_3(t)\end{aligned}$$

and therefore

$$\dot{V}_3(t) + 2\alpha V_3(t) = h\dot{x}^T(t)R\dot{x}(t) - 2\int_{t-h}^{t} e^{2\alpha(u-t)}\dot{x}^T(u)R\dot{x}(u)du. \tag{3.6}$$

By Lemma 2.2, we have

$$\begin{aligned}\int_{t-h}^{t} e^{2\alpha(u-t)}\dot{x}^T(u)R\dot{x}(u)du &\ge \frac{1}{\rho(\alpha)}(x(t)-x(t-h))^TR(x(t)-x(t-h))\\ &+\left(\frac{\eta_\alpha}{\sqrt{\beta_\alpha}}\right)^2\left[x(t) + \left(\frac{1}{l\eta_\alpha}-1\right)x(t-h) - \frac{1}{l\eta_\alpha}\int_{t-h}^{t}x(s)ds\right]^T\\ &\times R\left[x(t) + \left(\frac{1}{l\eta_\alpha}-1\right)x(t-h) - \frac{1}{l\eta_\alpha}\int_{t-h}^{t}x(s)ds\right],\\ &= \xi^T(t)\Bigg[\frac{1}{\rho(\alpha)}(e_1-e_2)^TR(e_1-e_2)\\ &\quad+\left(\frac{\eta_\alpha}{\sqrt{\beta_\alpha}}\right)^2\left(e_1 + \left(\frac{1}{l\eta_\alpha}-1\right)e_2 - \frac{1}{l\eta_\alpha}e_3\right)^T\end{aligned}$$

$$\times R\left(e_1 + \left(\frac{1}{l\eta_\alpha} - 1\right) e_2 - \frac{1}{l\eta_\alpha} e_3\right)\Bigg]\xi(t).$$

Moreover, since

$$h\dot{x}^T(t)R\dot{x}(t) = \xi^T(t)h(e_1^T A_0^T + e_2^T A_1^T + e_3^T A_2^T)R(A_0e_1 + A_1e_2 + A_2e_3)\xi(t),$$

from (3.6), we obtain

$$\dot{V}_3(t) + 2\alpha V_3(t) \leq \xi^T(t)M_3\xi(t), \tag{3.7}$$

where

$$\begin{aligned}
M_1 &= \begin{bmatrix} M_{11}^{(1)} & A_1^T P_1 - \frac{1}{2}P_2 & M_{13}^{(1)} \\ * & 0 & \frac{1}{2}A_1^T P_2 \\ * & * & A_2^T P_2 \end{bmatrix}, \\
M_{11}^{(1)} &= A_0^T P_1 + P_1 A_0 + 2\alpha P_1 + P_2, \\
M_{13}^{(1)} &= A_2^T P_1 + \alpha P_2 + \frac{1}{2}A_0 P_2, \\
M_2 &= \begin{bmatrix} Q & 0 & 0 \\ * & -e^{-2\alpha h}Q & 0 \\ * & * & 0 \end{bmatrix}, \\
M_3 &= \begin{bmatrix} M_{11}^{(3)} & M_{12}^{(3)} & M_{13}^{(3)} \\ * & M_{22}^{(3)} & M_{23}^{(3)} \\ * & * & M_{33}^{(3)} \end{bmatrix}, \\
M_{11}^{(3)} &= hA_0^T R A_0 - \frac{2}{\rho(\alpha)}R - 2\gamma R, \\
M_{12}^{(3)} &= \frac{1}{2}hA_0^T R A_1 - \frac{1}{\rho(\alpha)}R - \gamma\nu R, \\
M_{13}^{(3)} &= \frac{1}{2}hA_0^T R A_2 + \gamma(\nu + 1)R, \\
M_{22}^{(3)} &= hA_1^T R A_1 - \frac{2}{\rho(\alpha)}R - 2\gamma\nu R, \\
M_{23}^{(3)} &= \frac{1}{2}hA_1^T R A_2 + \gamma\nu(\nu + 1)R, \\
M_{33}^{(3)} &= hA_2^T R A_2 - 2\gamma(\nu + 1)^2 R.
\end{aligned}$$

From the above estimations, we get

$$\dot{V}(x_t) + 2\alpha V(x_t) \leq \xi^T(t)(M_1 + M_2 + M_3)\xi(t) = \xi^T(t)M\xi(t).$$

By condition (3.1), we have $\xi^T(t)M\xi(t) \le 0$ for all $t \ge 0$. Thus, by taking integral both sides from 0 to t, the last inequality gives

$$\begin{aligned} V(x_t) &\le V(\phi)e^{-2\alpha t}, \\ &\le \Lambda\|\phi\|^2 e^{-2\alpha t}\ t \ge 0. \end{aligned} \tag{3.8}$$

We now show that the functional $V(x_t)$ is positive definite. Indeed, it follows from (3.3) that

$$\begin{aligned} V_1(t) &= x^T(t)\begin{bmatrix} P_1 & P_2 \end{bmatrix}\begin{bmatrix} x^T(t) & \int_{t-h}^t x^T(u)du \end{bmatrix}^T \\ &= \begin{bmatrix} x(t) \\ \int_{t-h}^t x(u)du \end{bmatrix}^T \begin{bmatrix} P_1 & \frac{1}{2}P_2 \\ \frac{1}{2}P_2^T & 0 \end{bmatrix} \begin{bmatrix} x(t) \\ \int_{t-h}^t x(u)du \end{bmatrix}. \end{aligned} \tag{3.9}$$

By Lemma 2.1, we have

$$\int_s^t e^{2\alpha(u-t)}\dot{x}^T(u)R\dot{x}(u)du \ge \frac{2\alpha}{e^{2\alpha h-1}}(x(t)-x(s))^T R(x(t)-x(s))$$

by which we can get

$$V_2(t) + V_3(t) \ge \int_{t-h}^t \begin{bmatrix} x(t) \\ x(s) \end{bmatrix}^T \begin{bmatrix} \frac{1}{\rho(\alpha)}R & -\frac{1}{\rho(\alpha)}R \\ -\frac{1}{\rho(\alpha)}R & e^{-2\alpha h}Q + \frac{1}{\rho(\alpha)}R \end{bmatrix} \begin{bmatrix} x(t) \\ x(s) \end{bmatrix} ds. \tag{3.10}$$

Combining (3.9) and (3.10) we then obtain

$$\begin{aligned} V(x_t) &\ge \begin{bmatrix} x(t) \\ \int_{t-h}^t x(u)du \end{bmatrix}^T D \begin{bmatrix} x(t) \\ \int_{t-h}^t x(u)du \end{bmatrix} \\ &\ge \lambda_{\min}(D)\|x(t)\|^2. \end{aligned} \tag{3.11}$$

Finally, from (3.8) and (3.11) we readily obtain

$$\|x(t)\| \le \sqrt{\frac{\Lambda}{\lambda_{\min}(D)}}\|\phi\|e^{-\alpha t},\ t \ge 0.$$

This shows that system (2.2) is GES with exponential decay rate α. The proof is completed.

Remark 3.1. *Typically, in the existing results, the functionals $V_k(t)$, $k = 1, 2, 3$, are symmetric and nonnegative, that is, the matrix P, Q and R are restricted to be symmetric positive definite matrices [4], [6], [9]. Different from those, in Theorem 3.1, the matrices P and Q are not required to be positive and definite.*

By modifying the functional (3.3), we have the following improved result.

Theorem 3.2. *For a given $\alpha > 0$, system (2.2) is GES with decay rate α if there exist matrices $P = \begin{bmatrix} P_{11} & P_{12} \\ * & P_{22} \end{bmatrix}$, $Q = \begin{bmatrix} Q_1 & Q_2 \end{bmatrix}$ with $Q_1 = Q_1^T$ and $Z_1, Z_2 \in \mathbb{S}_n^+$ such that*

$$K = \begin{bmatrix} K_{11} & K_{12} & K_{13} \\ * & K_{22} & K_{23} \\ * & * & K_{33} \end{bmatrix} < 0, \tag{3.12}$$

$$F = \begin{bmatrix} e^{-2\alpha h} Q_1 + \frac{1}{\rho(\alpha)} Z_1 & \frac{e^{-2\alpha h}}{2} Q_1 \\ * & \frac{1}{\rho(\alpha)} Z_2 \end{bmatrix} > 0, \tag{3.13}$$

$$G = \begin{bmatrix} G_{11} & G_{12} & 0 & 0 \\ * & G_{22} & G_{23} & G_{24} \\ * & * & G_{33} & G_{34} \\ * & * & * & G_{44} \end{bmatrix} > 0, \tag{3.14}$$

where

$$\begin{aligned}
K_{11} &= A_0^T P_{11} + P_{11} A_0 + P_{12} + P_{21} + 2\alpha P_{11} \\
&\quad + Q_1 + h A_0^T Z_1 A_0 + h Z_2 - \frac{2}{\rho(\alpha)} Z_1 - 2\gamma Z_1, \\
K_{12} &= A_1^T P_{11} - P_{12} + h A_0^T Z_1 A_1 + \frac{2}{\rho(\alpha)} Z_1 - 2\gamma\nu Z_1, \\
K_{13} &= A_0^T P_{12} + + P_{11} A_2 + P_{22} + 2\alpha P_{12} \\
&\quad + h A_0^T Z_1 A_2 + 2\gamma(\nu + 1) Z_1 + \frac{1}{2} Q_2, \\
K_{22} &= h A_1^T Z_1 A_1 - e^{-2\alpha h} Q_1 - 2\gamma\nu^2 Z_1 - \frac{2}{\rho(\alpha)} Z_1, \\
K_{23} &= A_1^T P_{12} - P_{22} + h A_1^T Z A_2 + 2\gamma\nu(\nu + 1) Z_1 - \frac{1}{2} e^{-2\alpha h} Q_2, \\
K_{33} &= A_2^T P_{12} + P_{12} A_2 + 2\alpha P_{22} + A_2^T Z_1 A_2 \\
&\quad - 2\gamma(\nu + 1)^2 Z_1 - \frac{2}{\rho(\alpha)} Z_2 - \frac{1}{\rho(\alpha)} Q_2, \\
G_{11} &= P_{11} + \frac{h}{\rho(\alpha)} Z_1, \\
G_{12} &= P_{21} - \frac{1}{\rho(\alpha)} Z_1, \\
G_{22} &= P_{22} + \frac{4e^{-2\alpha h}}{h} Q_1 + \frac{4}{h\rho(\alpha)} Z_1, \\
G_{23} &= -\frac{6e^{-2\alpha h}}{h^2} Q_1 - \frac{6}{h^2 \rho(\alpha)} Z_1 + \frac{2e^{-2\alpha h}}{h} Q_2,
\end{aligned}$$

$$G_{24} = -\frac{3e^{-2\alpha h}}{h^2} Q_2,$$
$$G_{33} = \frac{12e^{-2\alpha}}{h^3} Q_1 - \frac{3e^{-2\alpha h}}{h^2} Q_2 + \frac{12}{h^3 \rho(\alpha)} Z_1 + \frac{4}{\rho(\alpha)} Z_2,$$
$$G_{34} = -\frac{6e^{-2\alpha h}}{h^3} Z_2, \quad G_{44} = \frac{12}{h^3 \rho(\alpha)} Z_2.$$

Proof. Consider the following functional

$$W(x_t) = W_1(t) + W_2(t) + W_3(t), \tag{3.15}$$

where

$$W_1(t) = \eta^T(t) P \eta(t), \quad \eta(t) = \begin{bmatrix} x^T(t) & \int_{t-h}^t x^T(u) du \end{bmatrix}^T,$$
$$W_2(t) = \int_{t-h}^t e^{2\alpha(s-t)} x^T(s) Q \begin{bmatrix} x^T(s) & \int_s^t x^T(u) du \end{bmatrix}^T ds,$$
$$W_3(t) = \int_{t-h}^t \int_s^t e^{2\alpha(u-t)} \left[\dot{x}^T(u) Z_1 \dot{x}(u) + x^T(u) Z_2 x(u) \right] duds.$$

Similar to the proof of Theorem 3.1, we manipulate the derivative of $W_k(t)$, $k = 1, 2, 3$, along state trajectories of system (2.2). First, the derivative of $W_1(t)$ is given by

$$\begin{aligned} \dot{W}_1(t) + 2\alpha W_1(t) &= \dot{\eta}^T(t) P \eta(t) + \eta^T(t) P \dot{\eta}(t) + 2\alpha W_1(t) \\ &= \xi^T(t) N_1 \xi(t), \end{aligned} \tag{3.16}$$

where

$$N_1 = \begin{bmatrix} U_{11} & U_{12} & U_{13} \\ * & U_{22} & U_{23} \\ * & * & U_{33} \end{bmatrix},$$
$$U_{11} = A_0^T P_{11} + P_{11} A_0 + P_{12} + P_{21} + 2\alpha P_{11},$$
$$U_{12} = A_1^T P_{11} - P_{12},$$
$$U_{13} = A_0^T P_{12} + + P_{11} A_2 + P_{22} + 2\alpha P_{12},$$
$$U_{22} = 0, \quad U_{23} = A_1^T P_{12} - P_{22},$$
$$U_{33} = A_2^T P_{12} + P_{12} A_2 + 2\alpha P_{22}.$$

Next, the derivative of $W_2(t)$ is manipulated as

$$\begin{aligned} \dot{W}_2(t) + 2\alpha W_2(t) &= x^T(t) Q_1 x(t) - e^{-2\alpha h} x^T(t-h) Q_1 x(t-h) \\ &\quad - e^{-2\alpha h} x^T(t-h) Q_2 \int_{t-h}^t x(u) du \end{aligned}$$

$$+\int_{t-h}^{t} e^{2\alpha(s-t)} x^T(s) Q_2 [x(t) - x(s)] ds.$$

By utilizing Lemma 2.1 to estimate the term $\int_{t-h}^{t} e^{2\alpha(s-t)} x^T(s) Q_2 x(s) ds$ we then obtain

$$\dot{W}_2(t) + 2\alpha W_2(t) \leq \xi^T(t) N_2 \xi(t), \tag{3.17}$$

where

$$N_2 = \begin{bmatrix} Q_1 & 0 & \frac{1}{2}Q_2 \\ * & -e^{-2\alpha h}Q_1 & -\frac{1}{2}e^{-2\alpha h}Q_2 \\ * & * & -\frac{1}{\rho(\alpha)}Q_2 \end{bmatrix}.$$

The derivative of $W_3(t)$ is given by

$$\begin{aligned}\dot{W}_3(t) = {} & h\dot{x}^T(t) Z_1 \dot{x}(t) + hx^T(t) Z_2 x(t) \\ & - 2\int_{t-h}^{t} e^{2\alpha(u-t)} \dot{x}^T(u) Z_1 \dot{x}(u) du \\ & - 2\int_{t-h}^{t} e^{2\alpha(u-t)} x^T(u) Z_2 x(u) du - 2\alpha W_3(t).\end{aligned}$$

By utilizing Lemma 2.1 and Lemma 2.2 to bound the terms

$$\int_{t-h}^{t} e^{2\alpha(u-t)} x^T(u) Z_2 x(u) du$$

and

$$\int_{t-h}^{t} e^{2\alpha(u-t)} \dot{x}^T(u) Z_1 \dot{x}(u) du,$$

respectively, we obtain

$$\dot{W}_3(t) + 2\alpha W_3(t) \leq \eta^T(t) N_3 \eta(t), \tag{3.18}$$

where

$$N_3 = \begin{bmatrix} N_{11} & N_{12} & N_{13} \\ * & N_{22} & N_{23} \\ * & * & N_{33} \end{bmatrix},$$

$$\begin{aligned} N_{11} &= hA_0^T Z_1 A_0 + 2\alpha P_{11} - \frac{2}{\rho(\alpha)} Z_1 - 2\gamma Z_1 + hZ_2, \\ N_{12} &= hA_0^T Z_1 A_1 + \frac{2}{\rho(\alpha)} Z_1 - 2\gamma\nu Z_1, \\ N_{13} &= hA_0^T Z_1 A_2 + 2\gamma(\nu + 1) Z_1, \\ N_{22} &= hA_1^T Z_1 A_1 - 2\gamma\nu^2 Z_1 - \frac{2}{\rho(\alpha)} Z_1, \end{aligned}$$

$$N_{23} = hA_1^T Z A_2 + 2\gamma\nu(\nu+1)Z_1,$$
$$N_{33} = A_2^T Z_1 A_2 - 2\gamma(\nu+1)^2 Z_1 - \frac{1}{\rho(\alpha)} Z.$$

From the above estimations, we have

$$\begin{aligned}\dot{W}(x_t) + 2\alpha W(t) &\le \xi^T(t)(N_1 + N_2 + N_3)\xi(t)\\ &= \xi^T(t) K \xi(t),\end{aligned} \tag{3.19}$$

According to condition (3.12), we have

$$\dot{W}(x_t) + 2\alpha W(x_t) \le 0,\ t \ge 0,$$

which leads to

$$W(x_t) \le W(\phi) e^{-2\alpha t}, \quad t \ge 0$$

by taking integral both sides of the last inequality. In addition, it can be verified from (3.15) that there exists a positive scalar κ such that

$$W(x_t) \le \kappa \|x_t\|^2,\ t \ge 0.$$

Therefore,

$$W(x_t) \le \kappa \|\phi\|^2 e^{-2\alpha t}, \quad t \ge 0.$$

To finalize the proof of this theorem, we will show that the functional (3.15) is positive definite. First, it can be seen that

$$W_1(t) = \begin{bmatrix} x(t) \\ \int_{t-h}^t x(u)du \end{bmatrix}^T \begin{bmatrix} P_{11} & P_{12} \\ * & P_{22} \end{bmatrix} \begin{bmatrix} x(t) \\ \int_{t-h}^t x(u)du \end{bmatrix}. \tag{3.20}$$

On the other hand, by Lemma 2.1, we have

$$\int_s^t e^{2\alpha(u-t)} \dot{x}^T(u) Z_1 \dot{x}(u) du \ge \frac{2\alpha}{e^{2\alpha h} - 1}(x(t) - x(s))^T Z_1 (x(t) - x(s)),$$
$$\int_s^t e^{2\alpha(u-t)} x^T(u) Z_2 x(u) du \ge \frac{2\alpha}{e^{2\alpha h} - 1}\left(\int_s^t x(u)du\right)^T Z_2 \left(\int_s^t x(u)du\right),$$

which yield

$$W_2(t) + W_3(t) \ge$$
$$\int_{t-h}^t \begin{bmatrix} x(t) \\ x(s) \\ \int_s^t x(u)du \end{bmatrix}^T \begin{bmatrix} \frac{1}{\rho(\alpha)} Z_1 & -\frac{1}{\rho(\alpha)} Z_1 & 0 \\ * & e^{-2\alpha h} Q_1 + \frac{1}{\rho(\alpha)} Z_1 & \frac{1}{2} e^{-2\alpha h} Q_2 \\ * & * & \frac{1}{\rho(\alpha)} Z_2 \end{bmatrix} \begin{bmatrix} x(t) \\ x(s) \\ \int_s^t x(u)du \end{bmatrix} ds. \tag{3.21}$$

By combining (3.20) and (3.21), we obtain

$$W(x_t) = W_1(t) + W_2(t) + W_3(t)$$
$$\geq \begin{bmatrix} x(t) \\ \int_{t-h}^{t} x(u)du \\ \int_{t-h}^{t}\int_{s}^{t} x(u)duds \\ \int_{t-h}^{t}\int_{\theta}^{t}\int_{s}^{t} x(u)dud\theta ds \end{bmatrix}^T G \begin{bmatrix} x(t) \\ \int_{t-h}^{t} x(u)du \\ \int_{t-h}^{t}\int_{s}^{t} x(u)duds \\ \int_{t-h}^{t}\int_{\theta}^{t}\int_{s}^{t} x(u)dud\theta ds \end{bmatrix}.$$

Thus, by condition (3.14), there exists a positive scalar $\tilde{\kappa}$ such that

$$W(x_t) \geq \tilde{\kappa}\|x(t)\|^2.$$

This leads to

$$\|x(t)\| \leq \sqrt{\frac{\kappa}{\tilde{\kappa}}}\|\phi\|e^{-\alpha t},\ t \geq 0,$$

by which we can conclude the 2D system (2.2) is GES with prescribed decay rate α. The proof is completed.

Remark 3.2. *The result of Theorem 1 and Theorem 2 in [8] can be deduced as a special case of Theorem 3.1 and Theorem 3.2 in this paper by specifying $\alpha = 0$ and $A_2 = 0$.*

4. An illustrative example

Consider system (2.2) with the following data [6]

$$A_0 = \begin{bmatrix} 0.2 & 0 \\ 0.2 & 0.1 \end{bmatrix}, \quad A_1 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad A_2 = \begin{bmatrix} -1 & 0 \\ -1 & -1 \end{bmatrix}.$$

By utilizing some novel weighted integral inequalities and based on the symmetric Lyapunov-Krasovskii functional approach, improved exponential stability criteria were obtained in [5] and [6]. Table 1 gives comparative results of the decay rate α for various delay h. It is clear that our approach based on an asymmetric functional approach can deliver less conservative results in regard to decay.

Table 1. Decay rate α for various h

h	0.3	0.5	0.8	1.0	1.5	1.6	1.8
[6]	0.1047	0.3456	0.9817	1.1610	0.3379	0.2478	0.1078
[5]	0.1048	0.3488	1.1436	1.1707	0.3631	0.2739	0.1344
Thm. 3.1	0.1048	0.3492	1.1444	1.1710	0.3645	0.2766	0.1352
Thm. 3.2	0.1050	0.3502	1.1454	1.1745	0.3658	0.2801	0.1382

5. Conclusions

In this paper, the problem of exponential estimate has been developed for linear systems with distributed delay. Based on an asymmetric Lyapunov–Krasovskii functional approach, new delay-dependent conditions have been derived in terms of linear matrix inequalities which can be effectively solved by various computational tools.

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