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ON STABILIZATION OF DISCRETE-TIME 2-D SYSTEMS IN ROESSER MODEL

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Abstract. This paper focuses on the stabilization problem of discrete 2-D linear systems described by the Roesser model. A tractable stability condition in terms of linear matrix inequalities is first reformulated. Then, two controller design schemes are developed to formulate stabilization conditions via state-feedback controllers (SFCs) and event-triggered controllers (ETCs). Numerical examples with simulations are given to illustrate the effectiveness of the design conditions.
Keywords: 2-D systems, Roesser model, event-triggered control.

1. Introduction

Two-dimensional (2-D) systems form an important category of dynamical systems. The dynamic propagation of a 2-D system occurs in each of the two independent directions [1]. This type of system can be used to model a large number of physical processes and engineering systems [1], [2]. Typical examples of 2-D systems can be found in finite approximations of partial differential equations, which appear in heat exchanger or gas filtration models [3], [4]. According to a wide range of applications and interesting features, the theory of 2-D systems has received significant research attention in the past few decades. In particular, the problems of performance analysis and controller, observers or filters design have been extensively studied and developed for various types of 2-D systems.

In modeling practical phenomena, 2-D systems are described by various state-space models. Among those, Roesser model [5] and Fornasini–Marchesini local state-space model (FMLSS) [6] are the most widely studied in the literature. Unlike the FMLSS model, in Roesser model, two independent state vector, called the horizontal and the vertical state vectors, are used to describe dynamics of 2-D systems which prevails in modeling control processes governed by partial differential equations. Only in very special cases, some state-space models of 2-D systems can be mutually transformed to each other. However, for complex 2-D systems, they are often studied separately due to the nature of their structures and potential applications [7].

In this paper, we focus on the stabilization problem of a class of 2-D systems in the Roesser model. Specifically, we first quote a tractable stability condition in terms of linear matrix inequalities (LMIs). Then, two design schemes are developed for obtaining stabilization conditions via state-feedback controllers (SFCs) and event-triggered controllers (ETCs). Numerical examples with simulations to illustrate the effectiveness of the design conditions are presented.

2. A motivation example

Various thermal processes in chemical reactors, heat exchanger or pipe furnaces are described by the following Darboux-type equation [1]

$$\frac{\partial T(x,t)}{\partial x} + \frac{\partial T(x,t)}{\partial t} = -aT(x,t) + bu(x,t), \tag{2.1}$$

where $T(x,t)$ is an unknown function representing, for example, temperature at space $x \in [0, t_f]$ and time t, $u = u(x,t)$ is the control input and a, b are real constants.

For given (small) increments Δx and Δt, by a simple forward Euler approximation

$$\frac{\partial T(x,t)}{\partial x} \approx \frac{T(i,j) - T(i-1,j)}{\Delta x}, \quad \frac{\partial T(x,t)}{\partial t} \approx \frac{T(i,j+1) - T(i,j)}{\Delta t},$$

and by defining two-parameter functions $T(i,j) = T(i\Delta x, j\Delta t)$, $u(i,j) = u(i\Delta x, j\Delta t)$, equation (2.1) can be written as

$$T(i,j+1) = \left(1 - \frac{\Delta t}{\Delta x} - a\Delta t\right) T(i,j) + \frac{\Delta t}{\Delta x} T(i-1,j) + b\Delta t u(i,j). \tag{2.2}$$

We define the state variables $x^h(i,j) = T(i-1,j)$ and $x^v(i,j) = T(i,j)$. Then, system (2.2) can be represented as

$$\begin{bmatrix} x^h(i+1,j) \\ x^v(i,j+1) \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 1 \\ \frac{\Delta t}{\Delta x} & 1 - \frac{\Delta t}{\Delta x} - a\Delta t \end{bmatrix}}_{A} \begin{bmatrix} x^h(i,j) \\ x^v(i,j) \end{bmatrix} + \underbrace{\begin{bmatrix} 0 \\ b\Delta t \end{bmatrix}}_{B} u(i,j). \tag{2.3}$$

Equation (2.3) represents a 2-D system in the Roesser model. More general, in this paper, we study the stabilization problem for the following 2-D system

$$\begin{bmatrix} x^h(i+1,j) \\ x^v(i,j+1) \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} x^h(i,j) \\ x^v(i,j) \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} u(i,j), \tag{2.4}$$

where $x^h(i,j) \in \mathbb{R}^{n_h}$ and $x^v(i,j) \in \mathbb{R}^{n_v}$ are the horizontal and vertical state vectors, respectively, $u(i,j) \in \mathbb{R}^{n_u}$ is the control input and

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \quad B = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} \in \mathbb{R}^{n \times n_u},$$

are given real matrices.

3. Stability conditions

Consider the following 2-D Roesser system

$$\begin{bmatrix} x^h(i+1,j) \\ x^v(i,j+1) \end{bmatrix} = \underbrace{\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}}_{A} \begin{bmatrix} x^h(i,j) \\ x^v(i,j) \end{bmatrix}, \tag{3.1}$$

where $A \in \mathbb{R}^{n\times n}$, $n = n_h + n_v$, is a given real matrix. Let $x(i,j) = \begin{bmatrix} x^h(i,j) \\ x^v(i,j) \end{bmatrix}$ denote the over all state vector. The initial condition of (3.1) is determined by square summable sequences $x^h(0,j)$ and $x^v(i,0)$.

Definition 3.1. *System* (3.1) *is said to be globally asymptotically stable (GAS) if any solution $x(i,j)$ of* (3.1) *satisfies*

$$\lim_{r\to\infty} X_r \triangleq \lim_{r\to\infty} \left\{ \sup_{i+j=r} \|x(i,j)\| \right\} = 0.$$

The characteristic polynomial of system (3.1) is given by

$$P(z_1,z_2) = \det \begin{bmatrix} I_{n_h} - z_1 A_{11} & -z_1 A_{12} \\ -z_2 A_{21} & I_{n_v} - z_2 A_{22} \end{bmatrix}.$$

The following theorem gives a stability criterion for system (3.1) based on the characteristic polynomial.

Theorem 3.1 (see, [1])**.** *The 2-D Roesser system* (3.1) *is GAS if and only if $P(z_1,z_2) \neq 0$ for all (z_1,z_2) that belongs to the close disk $\overline{D} \triangleq \{(z_1,z_2) \in \mathbb{C}^2 : |z_1| \leq 1, |z_2| \leq 1\}$.*

Theorem 3.1 provides a necessary and sufficient algebraic condition for the stability of system (3.1). Unfortunately, this condition is not tractable for the design purpose as it raises an NP-hard condition with closed-loop parameter-dependent matrix A. We now recast the criterion derived in Theorem 3.1 into the following LMI condition.

Theorem 3.2. *System* (3.1) *is GAS if there exists a symmetric positive definite matrix $P = \mathrm{diag}(P^h, P^v) \in \mathbb{S}_n^+$ that satisfies the following LMI condition*

$$A^\top P A - P < 0. \tag{3.2}$$

Proof. Assume in contrary that there exists a $(z_1,z_2) \in \overline{D}$ such that

$$P(z_1,z_2) = \det \begin{bmatrix} I_{n_h} - z_1 A_{11} & -z_1 A_{12} \\ -z_2 A_{21} & I_{n_v} - z_2 A_{22} \end{bmatrix} = 0.$$

Then, there exists a nonzero vector $\chi \in \mathbb{R}^{n\times n}$ such that

$$\begin{bmatrix} I_{n_h} - z_1 A_{11} & -z_1 A_{12} \\ -z_2 A_{21} & I_{n_v} - z_2 A_{22} \end{bmatrix} \chi = 0.$$

Thus,

$$\chi = \mathbb{I}(z_1, z_2)A\chi,$$

where, for convenience, we denote the block matrix $\mathbb{I}(\alpha, \beta) = \mathrm{diag}(\alpha I_{n_h}, \beta I_{n_v})$.

Let $Q = P - A^\top PA > 0$ and χ^*, z^* be the transpose conjugate vectors of χ and $z = (z_1, z_2)$. Then, we have

$$\begin{aligned}\chi^* P\chi &= \chi^* A^\top \mathbb{I}(z_1^*, z_2^*)P\mathbb{I}(z_1, z_2)A\chi \\ &= \chi^* A^\top \mathbb{I}\left(|z_1|^2, |z_2|^2\right) PA\chi \\ &= \chi^* P\chi - \chi^* Q\chi - \chi^* A^\top \mathbb{I}\left(1 - |z_1|^2, 1 - |z_2|^2\right) PA\chi \\ &\leq \chi^* P\chi - \chi^* Q\chi. \end{aligned} \tag{3.3}$$

Since $Q > 0$, it follows from the last inequality in (3.3) that $\chi^* Q\chi = 0$ and hence $\chi = 0$. This yields a contradiction with $\chi \neq 0$. The proof is completed. □

4. Controller design

4.1. Conventional state-feedback controllers

Consider a 2-D system as presented in (2.4). We aim to design an SFC that makes the closed-loop system of (2.4) stable in the form

$$u(i, j) = Kx(i, j), \tag{4.1}$$

where $K \in \mathbb{R}^{n_u \times n}$ is the controller gain matrix which will be determined. The closed-loop system of (2.4) and (4.1) is obtained as

$$\begin{bmatrix} x^h(i+1, j) \\ x^v(i, j+1) \end{bmatrix} = \underbrace{(A + BK)}_{A_c} \begin{bmatrix} x^h(i, j) \\ x^v(i, j) \end{bmatrix}. \tag{4.2}$$

By Theorem 3.2 (condition (3.2)), the closed-loop system (4.2) is GAS if there exists a matrix $P = \mathrm{diag}(P^h, P^v) \in \mathbb{S}_+^n$ such that

$$(A + BK)^\top P (A + BK) - P < 0. \tag{4.3}$$

Denote $X = P^{-1}$. By pre- and post-multiplying with X, condition (4.3) holds if and only if

$$X (A + BK)^\top X^{-1} (A + BK) X - X < 0. \tag{4.4}$$

By the Schur complement lemma, condition (4.5) is equivalent to

$$\begin{bmatrix} -X & XA^\top + XK^\top B^\top \\ AX + BKX & -X \end{bmatrix} < 0. \tag{4.5}$$

Finally, we define the matrix variable $KX = Y \in \mathbb{R}^{n_u \times n}$, condition (4.5) is reduced to the following LMI

$$\begin{bmatrix} -X & XA^\top + Y^\top B^\top \\ AX + BY & -X \end{bmatrix} < 0. \tag{4.6}$$

In summary, we have the following result.

Theorem 4.1. *The 2-D system* (2.4) *is stabilizable via SFCs if there exist a symmetric positive definite matrix* $X = \mathrm{diag}(X^h, X^v) \in \mathbb{S}^n_+$ *and a real matrix* $Y \in \mathbb{R}^{n_u \times n}$ *that satisfy the LMI condition* (4.6)*. The controller gain is obtained as*

$$K = YX^{-1},$$

where (X, Y) *is a feasible solution of* (4.6)*.*

4.2. Event-triggered control

In this section, we address the design problem of an event-triggered controller (ETC) for 2-D systems in the form of (2.4). Assume that the vertical and the horizontal state vectors are updated at instants i_p, j_q, and the latest states will be denoted as $x^h(i_p, j)$ and $x^v(i, j_q)$, respectively. The errors $e^h(i,j)$ and $e^v(i,j)$ between states $x^h(i,j)$, $x^v(i,j)$ and the latest transmitted states $x^h(i_p, j)$, $x^v(i, j_q)$ are defined by

$$\begin{bmatrix} e^h(i,j) \\ e^v(i,j) \end{bmatrix} = \begin{bmatrix} x^h(i_p,j) \\ x^v(i,j_q) \end{bmatrix} - \begin{bmatrix} x^h(i,j) \\ x^v(i,j) \end{bmatrix}.$$

From the latest transmitted instants i_p and j_q, the next horizontal and vertical time instants i_{p+1}, j_{q+1} are determined respectively by

$$i_{p+1} = i_p + \min\nolimits_{s>0} \left\{ s = i - i_p \middle| e^{h\top}(i,j)\Psi_h e^h(i,j) > \alpha_h x^{h\top}(i,j)\Psi_h x^h(i,j) \right\} \tag{4.7}$$

and

$$j_{q+1} = j_q + \min\nolimits_{t>0} \left\{ t = j - j_q \middle| e^{v\top}(i,j)\Psi_v e^v(i,j) > \alpha_v x^{v\top}(i,j)\Psi_v x^v(i,j) \right\}, \tag{4.8}$$

where α_h and α_v are positive scalars and Ψ_h, Ψ_v are positive definite matrices, which will be determined. This means that the states $x^h(i,j)$ and $x^v(i,j)$ will not be transmitted within the rectangle region $[i_p, i_{p+1}) \times [j_q, j_{q+1})$. Between two consecutive transmission instants, the following condition holds

$$\begin{cases} e^{h\top}(i,j)\Psi_h e^h(i,j) \le \alpha_h x^{h\top}(i,j)\Psi_h x^h(i,j), \\ e^{v\top}(i,j)\Psi_v e^v(i,j) \le \alpha_v x^{v\top}(i,j)\Psi_v x^v(i,j. \end{cases} \tag{4.9}$$

An ETC for system (2.4) will be designed in the form

$$u(i,j) = K \begin{bmatrix} x^h(i_p,j) \\ x^v(i,j_q) \end{bmatrix}, \tag{4.10}$$

where $x^h(i_p, j)$, $x^v(i, j_q)$ are the latest transmitted states and triggering time instants i_p, j_q will be determined by an event generator. The closed-loop system governed by (2.4) and (4.10) is given by

$$\begin{bmatrix} x^h(i+1,j) \\ x^v(i,j+1) \end{bmatrix} = A_c \begin{bmatrix} x^h(i,j) \\ x^v(i,j) \end{bmatrix} + A_e \begin{bmatrix} e^h(i,j) \\ e^v(i,j) \end{bmatrix}, \tag{4.11}$$

where $A_c = A + BK$ and $A_e = BK$.

Theorem 4.2. *For given scalars $\alpha_h > 0$, $\alpha_v > 0$, the closed-loop system* (4.11) *is GAS if there exist positive definite matrices $P = \operatorname{diag}(P^h, P^v)$ and $\Psi = \operatorname{diag}(\Psi_h, \Psi_v)$ that satisfy the following condition*

$$\left[\begin{array}{cc|c} -P + \mathbb{I}(\alpha_h, \alpha_v)\Psi & 0 & A_c^\top P \\ * & -\Psi & A_e^\top P \\ \hline * & * & -P \end{array}\right] < 0. \tag{4.12}$$

Proof. We denote the vectors

$$e(i,j) = \begin{bmatrix} e^h(i,j) \\ e^v(i,j) \end{bmatrix}, \quad \xi(i,j) = \begin{bmatrix} x(i,j) \\ e(i,j) \end{bmatrix}$$

and consider the Lyapunov function $V(x) = x^\top P x$. The difference of $V(x)$ along state trajectory $x(i,j)$ of system (4.11) is obtained as

$$\begin{aligned} \Delta V(x(i,j)) &= x^{h\top}(i+1,j)P^h x^h(i+1,j) - x^{h\top}(i,j)P^h x^h(i,j) \\ &\quad + x^{v\top}(i,j+1)P^v x^v(i,j+1) - x^{v\top}(i,j)P^v x^v(i,j) \\ &= \xi^\top(i,j)\left(\begin{bmatrix} A_c^\top \\ A_e^\top \end{bmatrix} P \begin{bmatrix} A_c & A_e \end{bmatrix} - \begin{bmatrix} P & 0 \\ 0 & 0 \end{bmatrix}\right)\xi(i,j). \end{aligned} \tag{4.13}$$

According to triggering condition (4.9), we have

$$x^\top(i,j)\mathbb{I}(\alpha_h, \alpha_v)\Psi x(i,j) - e^\top(i,j)\Psi e(i,j) \geq 0.$$

Therefore,

$$\xi^\top(i,j) \operatorname{diag}(\mathbb{I}(\alpha_h, \alpha_v)\Psi, -\Psi)\xi(i,j) \geq 0.$$

In combination with (4.13), we have

$$\Delta V(x(i,j)) \leq \xi^\top(i,j)\left(\begin{bmatrix} A_c^\top \\ A_e^\top \end{bmatrix} P \begin{bmatrix} A_c & A_e \end{bmatrix} + \begin{bmatrix} -P + \mathbb{I}(\alpha_h, \alpha_v)\Psi & 0 \\ 0 & -\Psi \end{bmatrix}\right)\xi(i,j). \tag{4.14}$$

By Schur complement lemma, condition (4.12) holds if and only if

$$\begin{bmatrix} A_c^\top \\ A_e^\top \end{bmatrix} P \begin{bmatrix} A_c & A_e \end{bmatrix} + \begin{bmatrix} -P + \mathbb{I}(\alpha_h, \alpha_v)\Psi & 0 \\ 0 & -\Psi \end{bmatrix} < 0.$$

Thus, if condition (4.12) holds, then there exists a positive scalar c_0 such that $\Delta V(x(i,j)) \leq -c_0\|x(i,j)\|^2$. Similar to [8], we can conclude that the closed-loop system (4.11) is GAS. The proof is completed. □

By utilizing the stability conditions presented in Theorem 4.3, an ETC in the form of (4.10) can be obtained as in the following theorem.

Theorem 4.3. *System* (2.4) *is stabilizable via ETC* (4.10) *if for given positive scalars* α_h *and* α_v*, there exist positive definite matrices* $X = \mathrm{diag}(X^h, X^v)$*,* $\widetilde{\Psi} = \mathrm{diag}(\widetilde{\Psi}_h, \widetilde{\Psi}_v)$ *and a matrix* Y *of appropriate dimensions that satisfy the following condition*

$$\left[\begin{array}{cc|c} -X + \mathbb{I}(\alpha_h, \alpha_v)\widetilde{\Psi} & 0 & (AX+BY)^\top \\ * & -\widetilde{\Psi} & (BY)^\top \\ \hline * & * & -X \end{array}\right] < 0. \tag{4.15}$$

The controller gain matrix is obtained as

$$K = YX^{-1}.$$

Proof. Let $P = X^{-1}$. We define

$$\widetilde{\Psi} = X\Psi X, \quad KX = Y.$$

By pre- and post-multiplying with $\mathcal{D}_X = \mathrm{diag}(X, X, X)$, we have

$$\mathcal{D}_X \begin{bmatrix} A_c^\top \\ A_e^\top \end{bmatrix} = \begin{bmatrix} X(A^\top + K^\top B^\top) \\ XK^\top B^\top \end{bmatrix} = \begin{bmatrix} (AX+BY)^\top \\ (BY)^\top \end{bmatrix}.$$

Then, condition (4.15) can be obtained by similar lines in Theorem 4.1. The proof is completed. □

4.3. Simulations

In this section, we validate the design conditions given in Theorem 4.3 by numerical simulations. Consider system (2.3) with the parameters

$$\Delta t = 0.05, \quad \Delta x = 0.1, \quad a = 0.1, \quad b = 1.$$

With the specified parameters, simulation results given in Figure 1 shows that the open system (i.e. without control) is unstable.

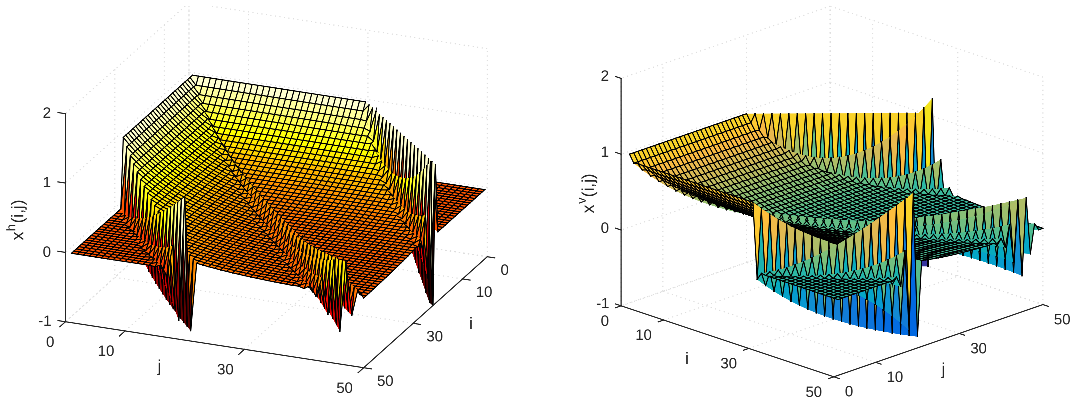
We now apply the proposed design conditions. By using the Matlab LMI toolbox, it is found that, for $\alpha_h = 0.8$ and $\alpha_v = 0.8$, condition (4.15) is feasible with the matrices

$$X = \mathrm{diag}(5.4721, 2.6291), \quad \widetilde{\Psi} = \mathrm{diag}(3.4317, 1.0380),$$
$$Y = \begin{bmatrix} -30.9179 & -17.7930 \end{bmatrix}.$$

By Theorem4.3, system (2.3) is stabilizable via the ETC (4.10). The controller gain is given by

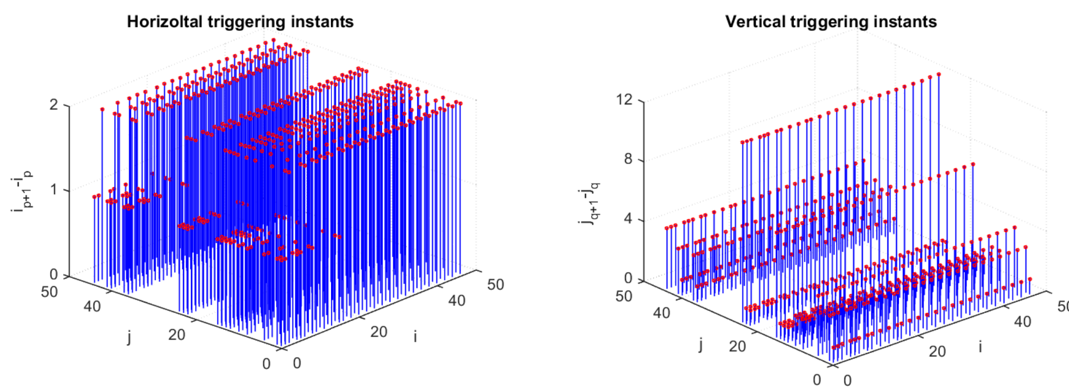
$$K = YX^{-1} = \begin{bmatrix} -5.6501 & -6.7678 \end{bmatrix}.$$

Simulation results with the ETC (4.10) are presented in Figures 2-4. It can be seen that directional ranges of triggering times are given in Figure 3, whereas 2-D triggering instants are plotted in Figure 4. State trajectories of the closed system are given in Figure 4. It is clear that the conducted closed trajectories are convergent as $i + j \to \infty$. This illustrate the effectiveness of the design method of Theorem 4.3.



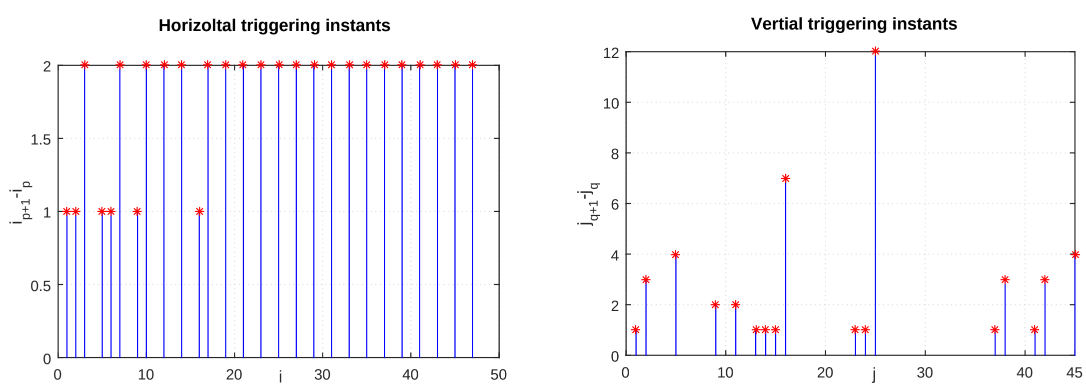
(a) $x^h(i,j)$ (b) $x^v(i,j)$

Figure 1. State trajectory of the open system



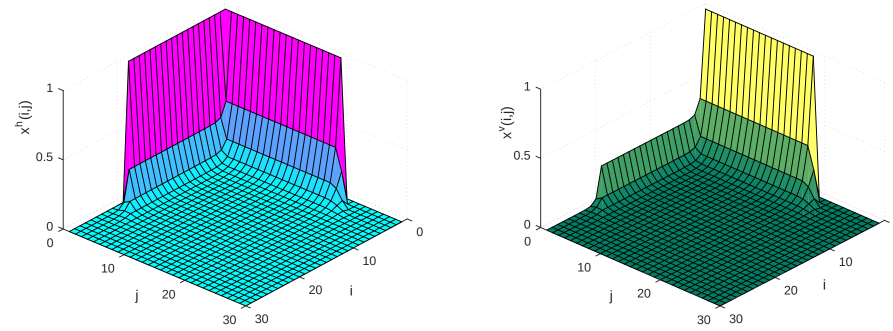
(c) $i_{p+1} - i_p$ (d) $j_{q+1} - j_q$

Figure 2. Triggering instants



(e) $i_{p+1} - i_p$ (f) $j_{q+1} - j_q$

Figure 3. 2-D triggering instants



(g) $x^h(i,j)$ (h) $x^v(i,j)$

Figure 4. Closed-loop state trajectory

5. Conclusions

In this paper, the problems of stability analysis and controller design have been addressed for discrete 2-D linear systems described by the Roesser model. Tractable design conditions for conventional state-feedback controller and event-triggered controllers have been derived. Numerical examples with simulations have been provided to validate the effectiveness of the proposed methods.

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