HNUE JOURNAL OF SCIENCE
Natural Science, 2025, Volume 70, Issue 3, pp. 26-34
This paper is available online at http://hnuejs.edu.vn/ns
DOI: 10.18173/2354-1059.2025-0034

A FIXED POINT THEOREM APPROACH TO GENERALIZED HALANAY INEQUALITIES

Le Anh Dung
Faculty of Mathematics, Hanoi National University of Education, Hanoi city, Vietnam
*Corresponding author: Le Anh Dung, e-mail: ladung@hnue.edu.vn

Received May 5, 2025. Revised May 30, 2025. Accepted June 16, 2025.

Abstract. In this paper, we aim to reformulate a generalized Halanay inequality. A new approach using fixed point theorems for contraction mappings is presented. The obtained estimation can be utilized in various problems for nonlinear time-delay systems such as stability, dissipativity analysis and control design.
Keywords: Halanay inequalities, exponential stability, fixed point theorem.

1. Introduction

Long-time behavior and, in particular, stability of time-delay systems are most important aspects in the theory of dynamical systems. In addition, in many practical models in science and engineering, dynamic of the system not only depends on the current states but also depend on the past states. These models are typically described by a certain type of dynamical systems with delays. Therefore, the study of time-delay systems is one of the most active areas in the past few decades [1]-[3].

In the stability analysis of dynamical systems, the Lyapunov functional method is a powerful technique. However, constructing a suitable and effective Lyapunov functional is not an easy task, especially for nonlinear systems. For nonlinear time-delay systems, an effective approach is the use of comparison techniques via differential inequalities. For example, in [1]-[5], various problems in systems and control theory have been investigated by utilizing the following Halanay-type inequality

$$D^+u(t) \leq \gamma(t) - \alpha(t)u(t) + \beta(t) \sup_{t-\tau(t)\leq s\leq t} u(s), \quad t \geq t_0.$$

In [6]-[7], generalized Halanay inequalities were utilized to study the stability of time-delay systems in neural networks and impulsive dynamics.

It is noted that in existing works, estimations for generalized Halanay inequalities have been formulated using the comparison principle, which is quite restrictive for nonautonomous and infinite dimensional systems. In this paper, we propose a new approach to estimations of generalized Halanay inequalities using fixed point theory.

2. Preliminaries

In order to discuss the stability of the zero solution of

$$D^+x(t) \leq -ax(t) + b \sup_{-\tau \leq s \leq 0} x(t+s),\ t \geq t_0, \tag{2.1}$$

where $\tau > 0$ is the time-delay and $D^+x(t)$ denotes the upper-right Dini derivative,

$$D^+x(t) = \limsup_{\epsilon \to 0^+} \frac{x(t+\epsilon) - x(t)}{\epsilon},$$

Halanay [8] proposed the following inequality.

Lemma 2.1 (Halanay inequality [8]). *For given $a > b \geq 0$, there exist scalars $\lambda > 0$ and $\kappa > 0$ such that*

$$x(t) \leq \kappa \|x_{t_0}\| e^{-\lambda(t-t_0)}, \quad t \geq t_0,$$

where $\|x_{t_0}\| = \sup_{-\tau \leq s \leq 0} x(t_0 + s)$.

Some variants of the inequality (2.1), called generalized Halanay inequalities with external perturbations, have been formulated for the following differential inequality

$$\begin{aligned} D^+x(t) &\leq -\lambda(t)x(t) + \delta(t) \sup_{-\tau(t) \leq s \leq 0} x(t+s) + \theta(t),\ t \geq t_0, \\ x(t) &= |\phi(t)|, \quad t \leq t_0, \end{aligned} \tag{2.2}$$

where $\tau : [0, \infty) \to [0, \infty)$ is a time-varying delay and $\lambda(t) \geq 0$, $\delta(t) \geq 0$, $\theta(t)$ are continuous functions.

Lemma 2.2 (see, [4]). *Let $x(t) \geq 0$ be a continuous functional satisfying the functional differential inequality* (2.2). *Assume that there exists a scalar $\sigma > 0$ such that*

$$-\lambda(t) + \delta(t) \leq -\sigma,\ t \geq t_0,$$

then the following inequality holds

$$x(t) \leq \frac{\|\theta\|_\infty}{\sigma} + \|\phi\|_\infty e^{-\eta(t-t_0)}, \quad t \geq t_0,$$

where

$$\|\phi\|_\infty = \sup_{s \leq t_0} |\phi(s)|, \quad \|\theta\|_\infty = \sup_{t \geq t_0} |\theta(t)|,$$

and the scalar $\eta > 0$ is defined as

$$\eta = \inf_{t \geq t_0} \left\{ \eta(t) : \eta(t) + \lambda(t) + \delta(t) e^{\eta(t)\tau(t)} = 0 \right\}.$$

In [3], the authors considered the problem of global generalized exponential stability of a class of nonautonomous functional differential equations based on the following result. To facilitate presenting the next result, let us define the following constants

$$\tau_{e\nu} = \inf\{\tau \geq t_0 : t - \tau(t) \geq t_0, \forall t \geq \tau\},$$
$$T_* = \inf\{T \geq \tau_{e\nu} : \sup_{t \geq T} \frac{\delta(t)}{\lambda(t)} < 1\},$$
$$\varrho = \sup_{t \geq T_*} \frac{\delta(t)}{\lambda(t)}, \quad I(\lambda) = \sup_{t \geq T_*} \int_{t-\tau(t)}^{t} \lambda(s)ds$$

and β_* is the unique positive solution of the scalar equation

$$H(\beta) = \beta + \varrho e^{\beta I(\lambda)} - 1 = 0.$$

We have the following result.

Lemma 2.3 (see, [3])**.** *Assume that*

($\mathbf{A1}$) $\lim_{t\to+\infty}(t - \tau(t)) = \infty$*;*

($\mathbf{A2}$) $\sup_{t\geq t_0} \int_{t-\tau(t)}^{t} \lambda(s)ds < \infty$ *and* $\lim_{t\to+\infty} \int_{t_0}^{t} \lambda(s)ds = \infty$*;*

($\mathbf{A3}$) $\sup_{t\geq t_0} \frac{\delta(t)}{\lambda(t)} < 1$.

Let $x(t) \geq 0$ *be a continuous function satisfying inequality* (2.2)*. Then, the following estimate holds*

$$x(t) \leq N\left(\|\phi\|_\infty - \frac{\theta_\lambda}{1-\delta_\infty^0}\right)^+ \exp\left(-\beta_* \int_{t_0}^{t} \lambda(s)ds\right) + \frac{\theta_\lambda}{1-\delta_\infty^0}, \quad t \geq t_0,$$

where, for a real number α*, the notation* α^+ *stands for* $\max\{\alpha, 0\}$ *and other constants are given as* $\theta_\lambda = \sup_{t\geq t_0} \frac{\theta(t)}{\lambda(t)}$, $\delta_\infty^0 = \sup_{t\geq t_0} \frac{\delta(t)}{\lambda(t)}$, $N = \exp\left(\beta_* \int_{t_0}^{T_*} \lambda(s)ds\right)$.

In addition, the authors of [5], [7] studied the exponential stability and boundedness of inequality (2.2) by integral inequalities and obtained the following result.

Lemma 2.4 (see, [7])**.** *Let assumptions* ($\mathbf{A1}$)-($\mathbf{A3}$) *hold and assume that there exists a constant* $M \geq 0$ *such that*

$$\int_{t_0}^{t} e^{-\int_s^t \lambda(u)du} \theta(s)ds \leq M.$$

Then, there exists a constant $\sigma \in (0, 1]$ *such that*

$$x(t) \leq \|\phi\|_\infty \exp\left\{-\sigma \int_{t_0}^{t} \lambda(s)ds\right\} + \frac{M}{1-\delta_\infty^0}, \quad t \in (-\infty, \infty).$$

Remark 2.1. *Lemma 2.2 is a proper extension of the result of Lemma 2.1 for the case of time-varying coefficients. However, it can be seen that Lemma 2.2 requires a restriction that the difference $\lambda(t) - \delta(t)$ is uniformly positive. In Lemmas 2.3 and 2.4, based on some new comparison techniques via differential and integral inequalities, this restriction was excluded. It should be pointed out that, in Lemmas 2.3, 2.4, the rate of change of the coefficients $\delta(t)$ and $\lambda(t)$ is still subject to some constraints. More specifically, although $\delta(t)$, $\lambda(t)$ can tend to zero, their ratio should be uniformly less than one. How to find alternative conditions to improve the results of Lemmas 2.3, 2.4 is of significant interest. This motivates us for the present study.*

In this paper, we propose a new approach to the above obtained results based on the fixed point theorem of contractive mappings.

3. Main results

In this section, the Banach fixed point theorem is employed to estimate the inequality (2.2). The obtained result is presented in the following theorem.

Theorem 3.1. *Let $x(t) \geq 0$ be a continuous function satisfying inequality* (2.2). *Assume that there exists a continuous function $h : [0, \infty) \to [0, \infty)$ and a constant $\beta \in [0, 1)$ such that*

$$|h(t) - \lambda(t)| + \delta(t) e^{\beta \int_{t-\tau(t)}^{t} h(s) ds} \leq (1 - \beta) h(t). \tag{3.1a}$$

$$\alpha = \sup_{t \geq t_0} \int_{t_0}^{t} e^{-\int_s^t h(\xi) d\xi} \left[|h(s) - \lambda(s)| + \delta(s) \right] ds < 1. \tag{3.1b}$$

$$\rho = \sup_{t \geq t_0} \int_{t_0}^{t} e^{-\int_s^t h(\xi) d\xi} \theta(s) ds < \infty. \tag{3.1c}$$

Then, it holds that

$$x(t) \leq \|\phi\|_\infty \exp\left\{ -\beta \int_{t_0}^{t} h(s) ds \right\} + \frac{\rho}{1 - \alpha}, \quad t \geq t_0.$$

Proof. It can be deduced by the comparison principle (see, e.g., [9], II. Theorem, p. 92) that

$$x(t) \leq X(t) \text{ for all } t \geq t_0,$$

where $X(t)$ is the unique solution of the functional differential equation

$$\begin{aligned} X'(t) &= -\lambda(t) X(t) + \delta(t) \sup_{-\tau(t) \leq s \leq 0} X(t+s) + \theta(t), \ t \geq t_0, \\ X(t) &= |\phi(t)|, \quad t \leq t_0. \end{aligned} \tag{3.2}$$

It follows from (3.2) that

$$X(t)=|\phi(t_0)|e^{-\int_{t_0}^t\lambda(s)ds}+\int_{t_0}^t e^{-\int_s^t\lambda(u)du}\delta(s)\sup_{s-\tau(s)\le u\le s}X(u)ds + \int_{t_0}^t e^{-\int_s^t\lambda(u)du}\theta(s)ds,\quad t\ge t_0.$$

Let $BC(-\infty,\infty)=\{x:\mathbb{R}\to\mathbb{R}: x \text{ is continuous and bounded}\}$. It is clear that $BC(-\infty,\infty)$ is a Banach space with the supremum norm $\|x\|=\sup_{t\in\mathbb{R}}|x(t)|$. We now define a subspace of $BC(-\infty,\infty)$ as follows:

$$\mathcal{S}=\left\{\psi\in BC(-\infty,\infty)\;\middle|\;\begin{array}{l}\psi_{|(-\infty,t_0]}=|\phi|,\\ \psi(t)\le\|\phi\|_\infty\exp\left\{-\beta\int_{t_0}^t h(s)ds\right\}+\dfrac{\rho}{1-\alpha},\ t\ge t_0\end{array}\right\}.$$

It can be verified that $\mathcal{S}$ is a closed subset of $BC(-\infty,\infty)$. Thus, $\mathcal{S}$ is also a complete metric space with induced norm of $BC(-\infty,\infty)$.

We define an operator $\Psi:\mathcal{S}\to\mathcal{S}$ by $(\Psi x)(t)=|\phi(t)|$ for $t\in(-\infty,t_0]$ and

$$\begin{aligned}(\Psi x)(t)=|\phi(t_0)|e^{-\int_{t_0}^t h(s)ds}&+\int_{t_0}^t e^{-\int_s^t h(\xi)d\xi}\theta(s)ds\\ &+\int_{t_0}^t e^{-\int_s^t h(\xi)d\xi}[h(s)-\lambda(s)]x(s)ds\\ &+\int_{t_0}^t e^{-\int_s^t h(\xi)d\xi}\delta(s)\sup_{s-\tau(s)\le\nu\le s}x(\nu)ds.\end{aligned}\tag{3.3}$$

First, we will show that $\Psi(\mathcal{S})\subset\mathcal{S}$. Indeed, for any $x\in\mathcal{S}$, by conditions (3.1a)-(3.1c), we have

$$\begin{aligned}(\Psi x)(t)&=|\phi(t_0)|e^{-\int_{t_0}^t h(\xi)d\xi}+\int_{t_0}^t e^{-\int_s^t h(\xi)d\xi}\theta(s)ds\\ &\quad+\int_{t_0}^t e^{-\int_s^t h(\xi)d\xi}\left[h(s)-\lambda(s)\right]x(s)ds\\ &\quad+\int_{t_0}^t e^{-\int_s^t h(\xi)d\xi}\delta(s)\sup_{s-\tau(s)\le\nu\le s}x(\nu)ds\\ &\le\rho+|\phi(t_0)|e^{-\int_{t_0}^t h(\xi)d\xi}\\ &\quad+\int_{t_0}^t e^{-\int_s^t h(\xi)d\xi}|h(s)-\lambda(s)|\left(\|\phi\|_\infty e^{-\beta\int_{t_0}^s h(\xi)d\xi}+\frac{\rho}{1-\alpha}\right)ds\\ &\quad+\int_{t_0}^t e^{-\int_s^t h(\xi)d\xi}\delta(s)\left(\|\phi\|_\infty e^{-\beta\int_{t_0}^{s-\tau(s)} h(\xi)d\xi}+\frac{\rho}{1-\alpha}\right)ds.\end{aligned}$$

Therefore,

$$
\begin{aligned}
&(\Psi x)(t) \leq \rho + |\phi(t_0)|e^{-\int_{t_0}^{t} h(\xi)d\xi} \\
&+ \frac{\rho}{1-\alpha}\int_{t_0}^{t} e^{-\int_{s}^{t} h(\xi)d\xi}\left(|h(s)-\lambda(s)|+\delta(s)\right)ds \\
&+ \|\phi\|_{\infty}\int_{t_0}^{t} e^{-\int_{s}^{t} h(\xi)d\xi}\left(|h(s)-\lambda(s)|e^{-\beta\int_{t_0}^{s} h(\xi)d\xi} + |\delta(s)|e^{-\beta\int_{t_0}^{s-\tau(s)} h(\xi)d\xi}\right)ds \\
&\leq \rho + |\phi(t_0)|e^{-\int_{t_0}^{t} h(\xi)d\xi} + \frac{\alpha\rho}{1-\alpha} \\
&+ \|\phi\|_{\infty}e^{-\int_{t_0}^{t} h(\xi)d\xi}\int_{t_0}^{t} e^{(1-\beta)\int_{t_0}^{s} h(\xi)d\xi}\left(|h(s)-\lambda(s)| + e^{\beta\int_{s-\tau(s)}^{s} h(\xi)d\xi}\delta(s)\right)ds \\
&\leq \rho + |\phi(t_0)|e^{-\int_{t_0}^{t} h(\xi)d\xi} + \frac{\alpha\rho}{1-\alpha} \\
&\quad + (1-\beta)\|\phi\|_{\infty}e^{-\int_{t_0}^{t} h(\xi)d\xi}\int_{t_0}^{t} e^{(1-\beta)\int_{t_0}^{s} h(\xi)d\xi}h(s)ds \\
&= \rho + |\phi(t_0)|e^{-\int_{t_0}^{t} h(\xi)d\xi} + \frac{\alpha\rho}{1-\alpha} + \|\phi\|_{\infty}e^{-\int_{t_0}^{t} h(\xi)d\xi}\left(e^{(1-\beta)\int_{t_0}^{t} h(\xi)d\xi} - 1\right) \\
&\leq \|\phi\|_{\infty}\exp\left\{-\beta\int_{t_0}^{t} h(s)ds\right\} + \frac{\rho}{1-\alpha}.
\end{aligned}
$$

The last inequality shows that $\Psi x \in \mathcal{S}$ for any $x \in \mathcal{S}$. Thus, $\Psi(\mathcal{S}) \subset \mathcal{S}$.

Next, we will show that Ψ is a contractive mapping. For this, let $x_1, x_2 \in \mathcal{S}$. It can be verified from (3.3) that

$$
\begin{aligned}
|(\Psi x_1)(t) - (\Psi x_2)(t)| &\leq \|x_1 - x_2\|_{\mathcal{S}}\int_{t_0}^{t} e^{-\int_{s}^{t} h(\xi)d\xi}\left(|h(s)-\lambda(s)|+\delta(s)\right)ds \\
&\leq \alpha\|x_1 - x_2\|_{\mathcal{S}}.
\end{aligned}
$$

Therefore,

$$
\|\Psi x_1 - \Psi x_2\|_{\mathcal{S}} = \sup_{t\geq t_0}|(\Psi x_1)(t) - (\Psi x_2)(t)| \leq \alpha\|x_1 - x_2\|_{\mathcal{S}}. \tag{3.4}
$$

This shows that Ψ is a contractive mapping. By the Banach fixed point theorem for contractive mappings, Ψ has a unique fixed point $x_* \in \mathcal{S}$, which is a solution of equation (3.2) with initial condition $x_*(t) = |\phi(t)|$ for $t \in (-\infty, t_0]$. Thus,

$$
X(t) \leq \|\phi\|_{\infty}\exp\left\{-\beta\int_{t_0}^{t} h(s)ds\right\} + \frac{\rho}{1-\alpha},\ t \geq t_0.
$$

The proof is completed. □

Remark 3.1. *It is worth noting that the derived conditions Theorem 3.1 improve most existing conditions for an exponential estimate of Halanay-type inequalities in the form of* (2.2)*. For example, if $\lambda(t)$ and $\delta(t)$ satisfy assumptions* (A2) *and* (A3) *then, by specifying $h(t) = \lambda(t)$, condition* (3.1a) *is reduced to*

$$\frac{\delta(t)}{\lambda(t)} e^{\beta \int_{t-\tau(t)}^{t} \lambda(s)ds} \leq 1 - \beta.$$

It is clear that

$$\frac{\delta(t)}{\lambda(t)} e^{\beta \int_{t-\tau(t)}^{t} \lambda(s)ds} \leq \delta_\infty^0 e^{\beta I_\lambda} < 1 - \beta$$

for any sufficiently small $\beta > 0$, where $I_\lambda = \sup_{t\geq t_0} \int_{t-\tau(t)}^{t} \lambda(s)ds < \infty$. Thus, condition (3.1a) *is fulfilled.*

As a special case, if $h(s) = \lambda(s)$ satisfies conditions (3.1a)-(3.1c) in Theorem 3.1, we have the following result.

Corollary 3.1. *Let $x(t) \geq 0$ be a continuous function satisfying the inequality* (2.2)*. If there exists a constant $\beta \in [0, 1)$ such that*

$$\sup_{t\geq t_0} \frac{\delta(t)}{\lambda(t)} e^{\beta \int_{t-\tau(t)}^{t} \lambda(s)ds} \leq 1 - \beta. \tag{3.5a}$$

$$\alpha = \sup_{t\geq t_0} \int_{t_0}^{t} e^{-\int_s^t \lambda(\xi)d\xi} \delta(s)ds < 1. \tag{3.5b}$$

$$\rho = \int_{t_0}^{t} e^{-\int_s^t \lambda(\xi)d\xi} \theta(s)ds < \infty. \tag{3.5c}$$

Then, the following inequality holds

$$x(t) \leq \|\phi\|_\infty \exp\left\{-\beta \int_{t_0}^{t} \lambda(s)ds\right\} + \frac{\rho}{1-\alpha}, \quad t \geq t_0.$$

4. An application

Let us consider the following nonlinear nonautonomous time-delay system

$$x'(t) = F(t, x(t), x(t - \tau(t))), \quad t \geq t_0, \tag{4.1}$$

where $x(t) \in \mathbb{R}^n$ is the state vector and $F : [t_0, \infty) \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ is a continuous vector filed satisfying $F(t, 0, 0) = 0$ and

$$2\langle F(t, u, v), u\rangle \leq -\varphi(t)\|u\|^2 + \psi(t)\|v\|^2 \tag{4.2}$$

for all $t \geq t_0$, where $\varphi(t)$ and $\psi(t)$ are continuous and positive functions. We also assume that the function F satisfies conditions such that the associated initial value problem of (4.1) has a unique global solution for each initial function.

Let $x(t)$ be a solution of (4.1) and $z(t) = \|x(t)\|^2$. We have

$$\begin{aligned} z'(t) &= 2\langle x(t), F(t, x(t), x(t-\tau(t)))\rangle \\ &\leq -\varphi(t)\|x(t)\|^2 + \psi(t)\|x(t-\tau(t))\|^2 \\ &\leq -\varphi(t)z(t) + \psi(t) \sup_{t-\tau(t)\leq s\leq t} z(s). \end{aligned}$$

By the comparison principle, we have the following result.

Corollary 4.1. *If there exist a continuous function $\gamma : [0,\infty) \to [0,\infty)$ and a constant $\beta \in [0,1)$ such that*

$$|\gamma(t) - \varphi(t)| + \psi(t)e^{\beta\int_{t-\tau(t)}^{t}\gamma(s)ds} \leq (1-\beta)\gamma(t), \tag{4.3}$$

then, for any solution $x(t,\phi)$ of (4.1)*, it holds that*

$$\|x(t,\phi)\| \leq \|\phi\|_\infty \exp\left\{-\beta\int_{t_0}^{t}\gamma(s)ds\right\}, \quad t \geq t_0.$$

In particular, if

$$\gamma_* = \liminf_{t\to\infty}\frac{\gamma(t)}{t} > 0$$

then system (4.1) *is globally exponentially stable (GES).*

For example, we consider the following nonlinear time-delay system, which is referred to the Mackey-Glass model

$$\begin{aligned} x_1'(t) &= -ax_1(t) + bx_2(t) + \frac{cx_1(t-\tau(t))}{1+x_2^2(t-\tau(t))}, \\ x_2'(t) &= -ax_2(t) + bx_1(t) + \frac{cx_2(t-\tau(t))}{1+x_1^2(t-\tau(t))}, \end{aligned} \tag{4.4}$$

where $a > 0$ and b, c are real scalars.

Let $u = (u_1, u_2)$, $v = (v_1, v_2)$ then the function F in (4.1) is defined as

$$F(t,u,v) = \begin{pmatrix} -au_1 + bu_2 + \frac{cv_1}{1+v_2^2} \\ -au_2 + bu_1 + \frac{cv_2}{1+v_1^2} \end{pmatrix}.$$

It can be verified that

$$\begin{aligned} \langle F(t,u,v), u\rangle &= -a\|u\|^2 + 2bu_1u_2 + c\left(\frac{u_1v_1}{1+v_2^2} + \frac{u_2v_2}{1+v_1^2}\right) \\ &\leq -a\|u\|^2 + |b|\|u\|^2 + |c|\|u\|\|v\| \\ &\leq -(a - |b| - 1/2|c|)\|u\|^2 + 1/2|c|\|v\|^2. \end{aligned}$$

We select $\gamma(t) = \gamma$ as a constant then condition (4.3) holds if and only if

$$a > |b| + |c|.$$

Thus, if $a > |b| + |c|$ then system (4.4) is GES as well established result in the literature.

5. Conclusions

In this paper, a new generalized Halanay inequality has been formulated. Based on a new approach utilizing a type of Banach fixed point theorem or contraction mappings, we derived upper estimates for a class of generalized Halanay inequalities. The obtained results can be utilized for various problems in the systems and control theory of time-delay systems such as stability, dissipativity analysis, and control design of nonlinear time-delay systems.

REFERENCES

[1] Tian HJ, Fan LQ & Xiang JX, (2007). Numerical dissipativity of multistep methods for delay differential equations. *Applied Mathematics and Computation*, 188, 934-941.

[2] Liu B, Lu W & Chen T, (2011). Generalized Halanay inequalities and their applications to neural networks with unbounded time-varying delays. *IEEE Transactions on Neural Networks*, 22, 1508-1513.

[3] Hien LV, Phat VN & Trinh H, (2015). New generalized Halanay inequalities with applications to stability of nonlinear non-autonomous time-delay systems. *Nonlinear Dynamics*, 82, 1-13.

[4] Wen L, Yu Y & Wang W, (2008). Generalized Halanay inequalities for dissipativity of Volterra functional differential equations. *Journal of Mathematical Analysis and Applications*, 347, 169-178.

[5] Ruan D & Liu Y, (2019). Generalized Halanay inequalities with applications to generalized exponential stability and boundedness of time-delay systems. *Mathematical Problems in Engineering*, No. 2019, 1-7.

[6] Lu B, Jiang H, Abdurahman A & Hu C, (2016). Global generalized exponential stability for a class of nonautonomous cellular neural networks via generalized Halanay inequalities. *Neurocomputing*, 214, 1046-1052.

[7] Wu Q, Zhang H, Xiang L & Zhou J, (2012). A generalized Halanay inequality on impulsive delayed dynamical systems and its applications. *Chaos, Solitons & Fractals*, 45, 56-62.

[8] Halanay A, (1966). Differential Equations: Stability, Oscillations and Time Lags. *Academic Press, London*.

[9] Walter W, (1970). Differential and Integral Inequalities. *Springer-Verlag, Berlin*.