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EXISTENCE AND LONG-TIME BEHAVIOR OF SOLUTIONS TO A CLASS OF DOUBLY NONLINEAR PARABOLIC EQUATIONS INVOLVING WEIGHTED p-LAPLACIAN OPERATORS

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Abstract. In this paper, we study the existence, uniqueness, and long-time behavior of global weak solutions to a class of doubly nonlinear degenerate parabolic equations involving weighted p-Laplacian operators in bounded domains with homogeneous Dirichlet boundary conditions. The global existence and uniqueness of weak solutions are established by combining compactness arguments with the theory of monotone operators. Furthermore, we analyze the asymptotic dynamics of the system and prove the existence of a compact global attractor by employing the theory of global attractors in bi-spaces.
Keywords: doubly nonlinear parabolic equations, weighted p-Laplacian operators, global solution, global attractor, asymptotic *a priori* estimate method, compactness method, monotonicity method.

1. Introduction

This paper is devoted to the study of a doubly nonlinear parabolic equation involving a weighted p-Laplacian operator. Specifically, we forcus on the existence, uniqueness, and long-time behavior of weak solutions to the following problem

$$\begin{cases} \dfrac{\partial \beta(u)}{\partial t} - \text{div}\,(\sigma(x)|\nabla u|^{p-2}\nabla u) + f(u) = g(x), & x \in \Omega, t > 0, \\ u(x,t) = 0, & x \in \partial\Omega, t > 0, \\ \beta(u(x,0)) = \beta(u_0(x)), & x \in \Omega, \end{cases} \tag{1.1}$$

where $p \geqslant 2$ and $\Omega \subset \mathbb{R}^N$ (with $N \geqslant 1$) denotes a bounded domain with smooth boundary $\partial\Omega$. The coefficient σ, nonlinearities $\beta,\ f$ as well as the external force g are assumed to satisfy assumptions specified later in the paper.

An elliptic model with diffusivity $\sigma(\cdot)$ was considered in [1]. In the last few years, problem (1.1) has been studied in both semilinear case and quasilinear case in bounded domains [2]-[5]. There are some results in doubly nonlinear cases, see [6]-[8].

We denote

$L_{p,\sigma}u := -\text{div}\left(\sigma|\nabla u|^{p-2}\nabla u\right)$;
p' is the conjugate exponent of p, i.e. $1/p + 1/p' = 1$;
$p^*_\gamma := \dfrac{pN}{N-p+\gamma}$, for $\gamma \in \mathbb{R}^+$;
$\mathcal{I}[p,q]$ is the closed interval with the endpoints p, q.

The following hypotheses are imposed.

(H1) The weight function $\sigma \in L^1_{loc}(\Omega)$ satisfies $\liminf\limits_{x\to z} |x-z|^{-\alpha}\sigma(x) > 0$ for some $\alpha \in (0,p)$ and $z \in \overline{\Omega}$;

(H2) The initial data u_0 and $\beta(u_0)$ belong to $L^2(\Omega)$;

(H3) The source term $g \in L^2(\Omega)$;

(H4) The function $\beta : \mathbb{R} \to \mathbb{R}$ is increasing and locally Lipschitzian continuous with $\beta(0) = 0$ and is bounded by $|\beta(u)| \leqslant C|u| + \tilde{C}, \quad C, \tilde{C} > 0$;

(H5) The function $f \in C^1(\mathbb{R})$ and there exists positive constants $C_0, C_1, C_2, \tilde{C}_0$ such that

$$\text{sign}(u)f(u) \geqslant C_1|u|^{q-1} - C_0, \tag{1.2}$$
$$|f(u)| \leqslant C_2\left(|u|^{q-1} + 1\right) \quad \text{with } q > \sup(2,p). \tag{1.3}$$
$$f'(u) \geqslant -\tilde{C}_0, \quad \forall u \in \mathbb{R}. \tag{1.4}$$

(H6) There exists $C_3 > 0$ such that $\xi \mapsto f(\xi) + C_3\beta(\xi)$ is an increasing function.

(H7) $[1, p^*_\alpha) \cap \mathcal{I}[p', q'] \neq \emptyset$.

Note that from conditions (1.2) of $(H5)$ and $(H4)$, we have

$$\text{sign}(u)f(u) \geqslant C_1|\beta(u)|^{q-1} - C_0. \tag{1.5}$$

Notations

Let β be a continuous function satisfying $\beta(0) = 0$. For $t \in \mathbb{R}$, we define $\Psi(t) := \int\limits_0^t \beta(\tau)d\tau$. Then, the Legendre transform of Ψ is given by

$$\Psi^*(\tau) = \sup_{s\in\mathbb{R}}\{\tau s - \Psi(s)\}.$$

The remaining of this paper is organized as follows. In Section 2, we present preliminary facts related to function spaces and operators that will be used later. In Section 3, by employing monotonicity arguments and the compactness method, we establish the global existence of weak solutions to problem (1.1). In Section 4, we prove the existence of a compact global attractor in $L^2(\Omega)$ for the semigroup associated to problem (1.1) by showing the existence of a bounded absorbing set in $\mathcal{D}_0^{1,p}(\Omega,\sigma)\cap L^\infty(\Omega)$ and using the compactness of the embedding $\mathcal{D}_0^{1,p}(\Omega,\sigma) \hookrightarrow L^2(\Omega)$.

2. Preliminary results

For the analysis of problem (1.1), we employ the weighted Sobolev space $\mathcal{D}_0^{1,p}(\Omega,\sigma)$, defined as the normed space obtained by completing $C_0^\infty(\Omega)$ under the norm

$$\|u\|_{\mathcal{D}_0^{1,p}(\Omega,\sigma)} = \left(\int_\Omega \sigma(x)|\nabla u|^p dx\right)^{1/p}.$$

Additionally, we employ the space $\mathcal{D}_0^{2,2}(\Omega,\sigma)$ of all functions $u \in C^\infty(\Omega)$ satisfying

$$\int_\Omega \left|\mathrm{div}\,(\sigma(x)\nabla u)\right|^2 dx < \infty.$$

Then, $\mathcal{D}_0^{2,2}(\Omega,\sigma)$ is a Banach space with the norm

$$\|u\|_{\mathcal{D}_0^{2,2}(\Omega,\sigma)} := \left(\int_\Omega \left|\mathrm{div}\,(\sigma(x)\nabla u)\right|^2 dx\right)^{1/2}.$$

It can be verified that, under assumption $(H1)$, the embedding $\mathcal{D}_0^{2,2}(\Omega,\sigma) \hookrightarrow \mathcal{D}_0^{1,2}(\Omega)$ (see [5]).

We recall several compactness results presented in [4], which extend the analogous results established by Caldiroli and Musina [1] for the particular case when $p = 2$.

Proposition 2.1. *Let $\Omega \subset \mathbb{R}^N$, $N \geqslant 1$, be a bounded domain and suppose that σ satisfies $(H1)$. As a result, we obtain the following embeddings*

(i) $\mathcal{D}_0^{1,p}(\Omega,\sigma) \hookrightarrow W_0^{1,\gamma}(\Omega)$ continuously for all $1 \leqslant \gamma < \dfrac{pN}{N+\alpha}$;

(ii) $\mathcal{D}_0^{1,p}(\Omega,\sigma) \hookrightarrow L^r(\Omega)$ compactly for all $1 \leqslant r < p_\alpha^*$, where $p_\alpha^* = \dfrac{pN}{N-p+\alpha}$.

The next proposition describes some fundamental properties of the operator $L_{p,\sigma}$, and its proof is rather straightforward.

Proposition 2.2. *The nonlinear operator $L_{p,\sigma}$ defined on the weighted Sobolev space $\mathcal{D}_0^{1,p}(\Omega,\sigma)$ maps into dual space $\mathcal{D}_0^{-1,p'}(\Omega)$. This operator satisfies the following key properties:*

(i) Hemicontinuity, i.e.,

The map $\lambda \mapsto \langle L_{p,\sigma}(u+\lambda v), w\rangle$ *is continuous from* $\mathbb{R}$ *to* $\mathbb{R}$*, for all* $u, v, w \in \mathcal{D}_0^{1,p}(\Omega,\sigma)$*. (ii) Strong monotonicity (for* $p \geqslant 2$*), i.e.,*

There exists a constant $\delta > 0$ *such that for all* $u, w \in \mathcal{D}_0^{1,p}(\Omega,\sigma)$*, the inequality holds*

$$\langle L_{p,\sigma}u - L_{p,\sigma}w, u-w\rangle \geqslant \delta \left\|u-w\right\|^p_{\mathcal{D}_0^{1,p}(\Omega,\sigma)}.$$

3. Existence and uniqueness of global weak solutions

For convenience, we shall use the following notations:

$$\Omega_T = \Omega \times (0,T), S_T = \partial\Omega \times (0,T),$$
$$V = L^p(0,T;\mathcal{D}_0^{1,p}(\Omega,\sigma)) \cap L^q(\Omega_T) \cap L^\infty(\tau,T;L^\infty(\Omega)) \quad \forall \tau > 0,$$
$$V^* = L^{p'}(0,T;\mathcal{D}^{-1,p'}(\Omega,\sigma)) + L^{q'}(\Omega_T).$$

Definition 3.1. *A function* u *is said to be a weak solution of problem* (1.1) *on the interval* $(0,T)$ *if*

$$u \in V,\ \frac{\partial \beta(u)}{\partial t} \in V^*;$$
$$u_{|t=0} = u_0 \text{ a.e. in } \Omega;\ \beta(u(0)) = \beta(u_0);$$

and for all test functions $\eta \in V$*, the following integral identity holds*

$$\int_{\Omega_T}\left(\frac{\partial\beta(u)}{\partial t}\eta + \sigma(x)|\nabla u|^{p-2}\nabla u\nabla\eta + f(u)\eta\right)dxdt = \int_{\Omega_T} g\eta dxdt$$

.

Noting that if $\dfrac{\partial\eta}{\partial t} \in L^2(0,T;L^2(\Omega))$ and $\eta(T) = 0$ then

$$\int_{\Omega_T}\frac{\partial\beta(u)}{\partial t}\eta dxdt = -\int_{\Omega_T}\left(\beta(u)-\beta(u_0)\right)\frac{\partial\eta}{\partial t}dxdt.$$

Lemma 3.1. *[4, Lemma 3.3] Suppose* $\{u_n\}$ *is a bounded sequence in the space* $L^p(0,T;\mathcal{D}_0^{1,p}(\Omega,\sigma))$ *and* $\{u_n'\}$ *is bounded in* V^**. Assume that the hypotheses* $(H1)$ *and* $(H7)$ *are satisfied, then there exists a subsequence of* $\{u_n\}$ *that converges almost everywhere in* Ω_T*.*

The next lemma follows directly from the consequence of Young's inequality combined with the embedding $\mathcal{D}_0^{1,p}(\Omega,\sigma) \hookrightarrow L^{p^*_\alpha}(\Omega)$, with $p^*_\alpha = \frac{pN}{N-p+\alpha}$.

Lemma 3.2. *Assume condition* $(H3)$ *holds. Then, for any* $u \in \mathcal{D}_0^{1,p}(\Omega,\sigma)$*, the following estimate is satisfield*

$$\left|\int_\Omega gudx\right| \leqslant \epsilon\|u\|^p_{\mathcal{D}_0^{1,p}(\Omega,\sigma)} + C(\epsilon)\|g\|^2_{L^2(\Omega)}, \quad \forall\epsilon > 0.$$

For a solution u of (1.1), we deduce that $\dfrac{\partial\beta(u)}{\partial t} = -L_{p,\sigma}u - f(u) + g$, then $\dfrac{\partial\beta(u)}{\partial t} \in L^{p'}\left(0,T;\mathcal{D}^{-1,p'}(\Omega,\sigma)\right)$. Since $q > \max\{2,p\}$, it follows that $\beta(u) \in L^q(\Omega_T) + L^\infty(0,T;L^2(\Omega)) \subset L^{q'}\left(0,T;L^{q'}(\Omega) + \mathcal{D}^{-1,p'}(\Omega,\sigma)\right)$. Using the Aubin-Lions lemma, we infer that $\beta(u) \in C([0,T];L^{q'}(\Omega))$, which guarantees the meaningfulness of the initial condition in (1.1) .

Theorem 3.1. *Let conditions* $(H1)-(H7)$ *hold. Then, for any initial datum* $u_0 \in L^2(\Omega)$ *and given* $T > 0$*, the problem* (1.1) *admits a unique weak solution* u *satisfying* $u \in L^p(0,T;\mathcal{D}_0^{1,p}(\Omega,\sigma)) \cap L^\infty(\tau,T;\mathcal{D}_0^{1,p}(\Omega,\sigma) \cap L^\infty(\Omega))$ *for all* $\tau > 0$ *and* $\beta(u) \in L^q(\Omega_T) \cap L^\infty(0,T;L^2(\Omega))$.

Proof. **i) Existence.** The existence result is obtained by deriving *a priori* estimates. Starting with the function β, a sequence $\beta_\epsilon \in C^1(\mathbb{R})$ is constructed with the properties: $\epsilon \leqslant \beta'_\epsilon$, $\beta_\epsilon(0) = 0$, $\beta_\epsilon \to \beta$ in $C_{\mathrm{loc}}(\mathbb{R})$ and $|\beta_\epsilon| \leqslant |\beta|$.

We then construct an approximating sequence $(u_{0\epsilon})_{\epsilon>0}$ in $C_0^\infty(\Omega)$ such that $u_{0\epsilon} \to u_0$ almost everywhere in Ω and both norms $\|u_{0\epsilon}\|_{L^2(\Omega)}$ and $\|\beta_\epsilon(u_{0\epsilon})\|_{L^2(\Omega)}$ are uniformly bounded by a constant $C > 0$. Consider the approximate problem

$$\begin{cases} \dfrac{\partial\beta_\epsilon(u_\epsilon)}{\partial t} + L_{p,\sigma}u_\epsilon + f(u_\epsilon) = g(x), \quad \text{in } \Omega_T, \\ u_\epsilon = 0, \quad \text{in } S_T, \\ \beta_\epsilon(u_\epsilon)_{|t=0} = \beta_\epsilon(u_{0\epsilon}), \quad \text{in } \Omega. \end{cases} \tag{3.1}$$

To begin, we demonstrate the existence and uniqueness of a solution u_ϵ to equation (3.1) satisfying $u_\epsilon \in L^\infty(\Omega_T) \cap L^\infty(0,T;\mathcal{D}_0^{1,p}(\Omega,\sigma))$. Since the argument closely parallels that found in [9, Lemma 5], we shall outline the main steps rather than presenting the full proof. By fixing an integer $m > 0$, we proceed by introducing the function

$$f_m(u) = \begin{cases} f(u) & \text{if } |\beta(u)| \leqslant m \\ C_1(|\beta(u)|^{q-1} - m^{q-1})\operatorname{sign}(u) + f(\beta^{-1}(u)\operatorname{sign}(u)) & \text{otherwise.} \end{cases}$$

Then

$$\operatorname{sign}(u)f_m(u) \geqslant C_1|\beta_\epsilon(u)|^{q-1} - C_0.$$

Indeed, when $|\beta(u)| \leqslant m$ we can apply the characteristics of β_ϵ along with assumption (1.5) to obtain

$$\operatorname{sign}(u)f_m(u) = \operatorname{sign}(u)f(u) \geqslant C_1|\beta(u)|^{q-1} - C_0 \geqslant C_1|\beta_\epsilon(u)|^{q-1} - C_0,$$

In the case where $|\beta(u)| \geqslant m$, we note that $\operatorname{sign}(u)/\operatorname{sign}(\beta^{-1}(m\operatorname{sign}(u))) = 1$ and from the properties of β_ϵ, we obtain

$$\begin{aligned} \operatorname{sign}(u)f_m(u) \geqslant& C_1(|\beta(u)|^{q-1} - m^{q-1}) + C_1|\beta(\beta^{-1}(m\operatorname{sign}(u)))|^{q-1} - C_0 \\ \geqslant& C_1|\beta(u)|^{q-1} - C_0 \geqslant C_1|\beta_\epsilon(u)|^{q-1} - C_0. \end{aligned}$$

Let $\gamma \in [0,1]$ be given, we define $K(\gamma,.)$ such that for each w, $K(\gamma,w) = u_{\epsilon,\gamma}$, where $u_{\epsilon,\gamma}$ is the solution to

$$\begin{cases} \dfrac{\partial\beta_\epsilon(u_{\epsilon,\gamma})}{\partial t} + L_{p,\sigma}u_{\epsilon,\gamma} + \gamma f_m(w) = g(x), \quad \text{in } \Omega_T, \\ u_{\epsilon,\gamma} = 0 \quad \text{in } S_T, \\ \beta_\epsilon(\, u_{\epsilon,\gamma})_{|t=0} = \beta_\epsilon(\gamma u_{0\epsilon}) \quad \text{in } \Omega, \end{cases} \tag{3.2}$$

Then, the operator $K(\gamma,.)$ is compact from the space $L^p(0,T;\mathcal{D}_0^{1,p}(\Omega,\sigma))$ into itself.

Infact, for a given $w \in L^p(0,T;\mathcal{D}_0^{1,p}(\Omega,\sigma))$, one has a unique solution $u_{\epsilon,\gamma} \in L^p(0,T;\mathcal{D}_0^{1,p}(\Omega,\sigma))$ by using the theory of Ladyzenskaya et. al. [10, Chap. V]. Moreover, by following arguments similar to those presented in [9, Lemma 5], for each $\gamma \in [0,1]$, $K(\sigma,.)$ is a compact operator and the mapping $\gamma \mapsto K(\gamma,.)$ is continuous. In particular, we have $K(0,w) = u_{\epsilon,0} = 0$. As a result, the LeraySchauder fixed point theorem guarantees the existence of a fixed point, i.e., a solution $u_\epsilon \equiv u_{\epsilon,1} = K(1,w)$. Furthermore, applying similar arguments as in [9, Lemma 5], together with inequality (3.5), yields the estimate

$$\|\beta_\epsilon(u_\epsilon)\|_{L^\infty(0,T;L^\infty(\Omega))} \leqslant C(u_{0\epsilon}),$$

The constant $C(u_{0\epsilon})$ is positive and depends only on the initial condition $u_{0\epsilon}$. Consequently, for any $m \geqslant C(u_{0\epsilon})$, we have $f_m(u_\epsilon) = f(u_\epsilon)$, which confirms that u_ϵ indeed solves equation (3.1).

As for the uniqueness of the solution, it can be established by invoking [11, Theorem 3, p. 1095] provided that we can verify the regularity condition $\dfrac{\partial\beta_\epsilon(u_\epsilon)}{\partial t} \in L^2(0,T;L^2(\Omega))$.

Next, we provide necessary *a priori* estimates for completing the proof.

We proceed by multiplying the first equation in (3.1) with the test function $|\beta_\epsilon(u_\epsilon)|^k\beta_\epsilon(u_\epsilon)$ and by applying the growth assumption on f, together with (1.5) and the properties of β_ϵ, this leads us to the following estimate:

$$\frac{1}{k+2}\frac{d}{dt}\int_\Omega |\beta_\epsilon(u_\epsilon)|^{k+2}dx + C_4\int_\Omega |\beta_\epsilon(u_\epsilon)|^{k+q}dx \leqslant C_5\int_\Omega |\beta_\epsilon(u_\epsilon)|^{k+1}dx. \tag{3.3}$$

Define $y_{\epsilon,k}(t) = \|\beta_\epsilon(u_\epsilon)(t)\|_{L^{k+2}(\Omega)}$. By applying Hölder's inequality on (3.3), we arrive at the inequality

$$\frac{dy_{\epsilon,k}(t)}{dt} + \lambda_0 y_{\epsilon,k}^{q-1}(t) \leqslant \alpha_0;$$

Here $\alpha_0 > 0$ and $\lambda_0 > 0$ are constants. Utilizing Ghidaglias lemma as presented in [12], it follows that

$$y_{\varepsilon,k}(t) \leqslant \left(\frac{\alpha_0}{\lambda_0}\right)^{\frac{1}{q-1}} + \frac{1}{[\lambda_0(q-2)t]^{\frac{1}{q-2}}} = C_4(t)\,, \text{for all } t > 0. \tag{3.4}$$

As $k \to +\infty$, we deduce that for all $t \geqslant \tau > 0$,

$$\|\beta_\epsilon(u_\epsilon)(t)\|_{L^\infty(\Omega)} \leqslant C_4(\tau); \tag{3.5}$$

which implies that

$$\|u_\epsilon(t)\|_{L^\infty(\Omega)} \leqslant \max(\beta_\epsilon^{-1}(C_4(\tau)), |\beta_\epsilon^{-1}(-C_4(\tau))|) = \delta_\epsilon. \tag{3.6}$$

Given that β_ϵ converges to β in $C_{\text{loc}}(\mathbb{R})$,$\delta_\epsilon$ is bounded in $\mathbb{R}$ as $\epsilon \to +\infty$. Hence, we can assert that

$$\delta_\epsilon \leqslant \max(\beta^{-1}(C_4(\tau)), |\beta^{-1}(-C_4(\tau))|).$$

As a result, we arrive at the uniform estimate

$$\|u_\epsilon\|_{L^\infty(\tau,T;L^\infty(\Omega))} \leqslant C_6(\tau, T). \tag{3.7}$$

Additionally, setting $k = 0$ in (3.3) and applying Hölder's inequality followed by integration over the interval $[0, T]$ leads to

$$\|\beta_\epsilon(u_\epsilon)\|_{L^\infty(0,T;L^2(\Omega))\cap L^q(\Omega_T)} \leqslant C_7(T). \tag{3.8}$$

Multiply the first equation in (3.1) by u_ϵ, then integrate over Ω. Making use of assumption (1.2), Lemma 3.2, $(H3)$ and the known characteristics of β_ϵ, gives

$$\frac{d}{dt}\left(\int_\Omega \Psi_\epsilon^*(\beta_\epsilon(u_\epsilon))dx\right) + \int_\Omega \sigma(x)|\nabla u_\epsilon|^p dx + C_1 \int_\Omega |u_\epsilon|^q dx \leqslant C(C_0, |\Omega|, \|g\|^2_{L^2(\Omega)}), \tag{3.9}$$

Here, Ψ_ϵ^* denotes the Legendre transform of Ψ_ϵ, defined by $\Psi_\epsilon(t) = \int_0^t \beta_\epsilon(s)ds$. From assumptions $(H1)$ and $(H2)$, and since $\Psi^*(\beta(u_0)) \in L^1(\Omega)$, we know that $\int_\Omega \Psi_\epsilon^*(\beta_\epsilon(u_{0\epsilon}))dx$ converges to $\int_\Omega \Psi^*(\beta(u_0))dx \leqslant C$ with $C > 0$. Therefore,

$$\int_\Omega \Psi_\epsilon^*\left(\beta_\epsilon(u_\epsilon)\right)dx + C_8 \int_0^T \int_\Omega \sigma(x) \mid \nabla u_\epsilon|^p \, dxds + C_1 \int_0^T \int_\Omega |u_\epsilon|^q dxds \leqslant C_8(T). \tag{3.10}$$

Hence

$$\|u_\epsilon\|_{L^p(0,T;\mathcal{D}_0^{1,p}(\Omega,\sigma))} \leqslant C_8(T). \tag{3.11}$$

By combining (3.1) with (3.6), we get for any $t \geqslant \tau > 0$,

$$\int_\Omega \beta_\epsilon'(u_\epsilon)(\frac{\partial u_\epsilon}{\partial t})^2 dx + \frac{d}{dt}\left[\frac{1}{p}\int_\Omega \sigma(x)|\nabla u_\epsilon|^p dx + \int_\Omega \int_0^{u_\epsilon} f(y)dy\, dx - \int_\Omega g u_\epsilon dx\right] = 0\,. \tag{3.12}$$

By integrating (3.9) on $[t, t+\frac{\tau}{2}]$, we get

$$\int_t^{t+\frac{\tau}{2}} \left(\int_\Omega \sigma(x) |\nabla u_\epsilon|^p dx + C_1 \int_\Omega |u_\epsilon|^q dx \right) dt \leqslant C_9(\tau), \quad \forall t \geqslant \frac{\tau}{2}.$$

Additionally, making use of (1.3), we obtain

$$\tilde{C}_1 \int_\Omega |u_\epsilon|^q dx - \tilde{C}_0 \leqslant \int_\Omega \int_0^{u_\epsilon(x,t)} f(y) dy\, dx \leqslant \tilde{C}_1 \int_\Omega |u_\epsilon|^q dx + \tilde{C}_0. \tag{3.13}$$

So, using Lemma 3.2 , we infer that

$$\int_t^{t+\frac{\tau}{2}} \left[\int_\Omega \sigma(x) |\nabla u_\epsilon|^p dx + \int_\Omega \int_0^{u_\epsilon(x,t)} f(y) dy\, dx - \int_\Omega g u_\epsilon dx \right] dt \leqslant C_9(\tau), \forall t \geqslant \frac{\tau}{2}. \tag{3.14}$$

Then, applying the uniform Gronwall inequality with

$$y(t) = \int_\Omega \sigma(x) |\nabla u_\epsilon|^p dx + \int_\Omega \int_0^{u_\epsilon(x,t)} f(y) dy\, dx - \int_\Omega g u_\epsilon dx,$$

gives

$$\int_\Omega \sigma(x) |\nabla u_\epsilon|^p dx + \int_\Omega \int_0^{u_\epsilon(x,t)} f(y) dy\, dx - \int_\Omega g u_\epsilon dx \leqslant C_{10}(\tau), \quad \forall t \geqslant \tau > 0. \tag{3.15}$$

By applying (3.13), (3.15) and Lemma 3.2 we arrive at

$$\int_\Omega \sigma(x) |\nabla u_\epsilon(t)|^p dx + \int_\Omega |u_\epsilon(t)|^q dx \leqslant C_{11}(\tau), \forall t \geqslant \tau > 0. \tag{3.16}$$

Hence

$$\|u_\epsilon\|_{L^\infty(\tau,T;\mathcal{D}_0^{1,p}(\Omega,\sigma))} \leqslant C_{11}(\tau, T). \tag{3.17}$$

Next, by integrating (3.12) over $[\tau, T]$ and utilizing both (3.6) and (3.15) , we can conclude that

$$\int_\tau^T \int_\Omega \beta'_\epsilon(u_\epsilon) (\frac{\partial u_\epsilon}{\partial t})^2 dx ds \leqslant C_{13}(\tau, T). \tag{3.18}$$

By integrating (3.12) on $[t, t+\tau]$ and using (3.6), (3.15) again, it follows that

$$\int_t^{t+\tau} \int_\Omega \beta'_\epsilon(u_\epsilon) (\frac{\partial u_\epsilon}{\partial t})^2 dx ds \leqslant C_{14}(\tau), \text{ for any } t \geqslant \tau > 0. \tag{3.19}$$

We denote by L the Lipschitz constant of β on $[-\delta, \delta]$, here, δ is the bound in (3.6). Choose β_ϵ such that $0 < \beta_\epsilon' \leqslant L$ on $[-\delta, \delta]$. From (3.18) and (3.19), we infer that

$$\frac{1}{L}\int_\tau^T \int_\Omega \left(\frac{\partial \beta_\epsilon(u_\epsilon)}{\partial t}\right)^2 dxds \leqslant C_{15}(\tau, T), \text{ for any } T \geqslant \tau > 0,$$

$$\frac{1}{L}\int_t^{t+\tau} \int_\Omega \left(\frac{\partial \beta_\epsilon(u_\epsilon)}{\partial t}\right)^2 dxds \leqslant \tilde{C}_{15}(\tau, T), \text{ for any } \tau > 0.$$

This implies that

$$\int_\tau^T \int_\Omega \left(\frac{\partial \beta_\epsilon(u_\epsilon)}{\partial t}\right)^2 dx\, ds \leqslant C_{16}(\tau, T), \quad \text{for } T \geqslant \tau > 0, \tag{3.20}$$

$$\text{and} \int_t^{t+\tau} \int_\Omega \left(\frac{\partial \beta_\epsilon(u_\epsilon)}{\partial t}\right)^2 dx\, ds \leqslant C_{17}(\tau), \quad \text{for } \tau > 0. \tag{3.21}$$

Passage to the limit in (3.1) as $\epsilon \to +\infty$.

Using Hölder's inequality, we have

$$\begin{aligned}\Big| \int_0^T \langle L_{p,\sigma} u, w\rangle \, dt\Big| &= \Big| \int_0^T \int_\Omega \sigma(x)|\nabla u_\epsilon|^{p-2}\nabla u_\epsilon \nabla w dxdt\Big| \\ &\leqslant \int_0^T \int_\Omega \left(\sigma(x)^{\frac{p-1}{p}}|\nabla u_\epsilon|^{p-1}\right)\left(\sigma(x)^{\frac{1}{p}}|\nabla w|\right) dxdt \\ &\leqslant \|u_\epsilon\|^{\frac{p}{p'}}_{L^p(0,T;\mathcal{D}_0^{1,p}(\Omega,\sigma))}\|w\|_{L^p(0,T;\mathcal{D}_0^{1,p}(\Omega,\sigma))},\end{aligned} \tag{3.22}$$

for any $w \in L^p(0, T; \mathcal{D}_0^{1,p}(\Omega, \sigma))$. From (3.11) and (3.7), we have the boundedness of $\{u_\epsilon\}$ in $L^p(0, T; \mathcal{D}_0^{1,p}(\Omega, \sigma))$. So

$$L_{p,\sigma} u_\epsilon \text{ is bounded in } L^{p'}(0, T; \mathcal{D}^{-1,p'}(\Omega, \sigma)). \tag{3.23}$$

By (3.20) and (3.21), we have

$$\frac{\partial \beta_\epsilon(u_\epsilon)}{\partial t} \text{ is bounded in } L^2(\tau, T; L^2(\Omega)), \forall \tau > 0. \tag{3.24}$$

Thanks to estimates (3.5),(3.7),(3.8), (3.11), (3.17), (3.23) and Lemma 3.1, let us consider a subsequence, again denoted by u_ϵ), for which the following convergence properties are satisfied as $\epsilon \to +\infty$

$$u_\epsilon \rightharpoonup u \quad \text{weakly in } L^p(0, T; \mathcal{D}_0^{1,p}(\Omega, \sigma)), \tag{3.25}$$

$$u_\epsilon \rightharpoonup u \quad \text{weakly star in } L^\infty(\tau, T; \mathcal{D}_0^{1,p}(\Omega)), \quad \forall \tau > 0, \tag{3.26}$$

$$\beta_\epsilon(u_\epsilon) \rightharpoonup \zeta \quad \text{weakly in } L^q(\Omega_T), \tag{3.27}$$

$$\beta_\epsilon(u_\epsilon) \rightharpoonup \zeta \quad \text{weakly star in } L^\infty(\tau, T; L^\infty(\Omega)), \tag{3.28}$$

$$L_{p,\sigma} u_\epsilon \rightharpoonup \chi \quad \text{weakly in } L^{p'}(0, T; \mathcal{D}^{-1,p'}(\Omega, \sigma)). \tag{3.29}$$

Referring to (3.6), (3.24), (3.27), (3.28), and applying the Aubin-Lions lemma, we conclude that $\beta_\epsilon(u_\epsilon) \to \zeta$ strongly in $C([0,T], L^2(\Omega))$. Using an argument similar to that developed in ([13, p. 1048]), it follows that $\beta(u) = \zeta$. To establish that u is a weak solution of (1.1), it suffics to note, following [9, p. 108], that $f(u_\epsilon) \to f(u)$ strongly in $L^1(\Omega_T)$ and in $L^s(\tau, T; L^s(\Omega))$ for any $\tau > 0$ and $s \geqslant 1$. This follows from the growth conditions imposed on the regularized function f_ϵ along with an application of Vitali's theorem.

The equation (3.1) can be reformulated in V^* as follows

$$\frac{\partial \beta_\epsilon(u_\epsilon)}{\partial t} = g - L_{p,\sigma} u_\epsilon - f(u_\epsilon).$$

Hence

$$\frac{\partial \beta(u)}{\partial t} = g - \chi - f(u). \tag{3.30}$$

Next, we aim to demonstrate that $\chi = L_{p,\sigma} u$. We have

$$X_\epsilon := \int_0^T \langle L_{p\sigma} u_\epsilon - L_{p,\sigma} w, u_\epsilon - w \rangle \, dt \geqslant 0, \tag{3.31}$$

for every $w \in L^p(0,T; \mathcal{D}_0^{1,p}(\Omega, \sigma))$. Noting that

$$\begin{aligned} \int_0^T \langle L_{p,\sigma} u_\epsilon, u_\epsilon \rangle \, dt &= \int_0^T \int_\Omega \sigma(x) |\nabla u_\epsilon|^p dx dt \\ &= \int_0^T \int_\Omega (g(x) u_\epsilon - f(u_\epsilon) u_\epsilon - \frac{\partial \beta_\epsilon(u_\epsilon)}{\partial t} u_\epsilon) dx dt \\ &= \int_0^T \int_\Omega (g(x) u_\epsilon - f(u_\epsilon) u_\epsilon) dx dt \\ &\quad + \int_\Omega \Psi^*(\beta_\epsilon(u_\epsilon(0))) dx - \int_\Omega \Psi^*(\beta_\epsilon(u_\epsilon(T))) dx. \end{aligned} \tag{3.32}$$

Therefore,

$$\begin{aligned} X_\epsilon = &\int_0^T \int_\Omega (g(x) u_\epsilon - f(u_\epsilon) u_\epsilon) dx dt + \int_\Omega \Psi^*(\beta_\epsilon(u_\epsilon(0))) dx - \int_\Omega \Psi^*(\beta_\epsilon(u_\epsilon(T))) dx \\ &- \int_0^T \langle L_{p,\sigma} u_\epsilon, w \rangle \, dt - \int_0^T \langle L_{p,\sigma} w, u_\epsilon - w \rangle \, dt. \end{aligned} \tag{3.33}$$

From the definition of $\Psi(\beta_\epsilon(u_\epsilon(0))$, it follows that $\Psi(\beta_\epsilon(u_\epsilon(0)) \to \Psi(\beta(u_0)$. Additionally, by the lower semicontinuity of $\| \cdot \|_{L^2(\Omega)}$, we obtain

$$\int_\Omega \Psi^*(\beta(u(T))) dx \leqslant \lim_{\epsilon \to +\infty} \inf \int_\Omega \Psi^*(\beta_\epsilon(u_\epsilon(T))) dx. \tag{3.34}$$

At the same time, by the Lebesgue dominated theorem, we have

$$\int_0^T \int_\Omega (g(x)u - f(u)u)dxdt = \lim_{\epsilon\to+\infty} \int_0^T \int_\Omega (g(x)u_\epsilon - f(u_\epsilon)u_\epsilon)dxdt.$$

This observation, combining with (3.30), (3.34), we have

$$\begin{aligned}\limsup_{\epsilon\to\infty} X_\epsilon \leqslant & \int_0^T \int_\Omega (g(x)u - f(u)u)dxdt + \int_\Omega \Psi^*(\beta(u_0))dx - \int_\Omega \Psi^*(\beta(u(T)))dx \\ & - \int_0^T \langle \chi, w\rangle\, dt - \int_0^T \langle L_{p,\sigma} w, u - w\rangle\, dt.\end{aligned} \tag{3.35}$$

Based on the result in (3.30), we obtain

$$\int_0^T \int_\Omega (g(x)u - f(u)u)dxdt + \int_\Omega \Psi^*(\beta(u_0))dx - \int_\Omega \Psi^*(\beta(u(T)))dx = \int_0^T \langle \chi, u\rangle\, dt.$$

This equation and (3.35) deduce that

$$\int_0^T \langle \chi - L_{p,\sigma} w, u - w\rangle\, dt \geqslant 0. \tag{3.36}$$

We choose $w = u - \lambda v$, where $v \in L^p(0, T; \mathcal{D}_0^{1,p}(\Omega, \sigma))$ and $\lambda > 0$. Making use of (3.36), we arrive at the following estimate

$$\int_0^T \langle \chi - L_{p,\sigma}(u - \lambda v), v\rangle\, dt \geqslant 0.$$

Passing to the limit as $\lambda \to 0$ and employing the hemicontinuity of the operator $L_{p,\sigma}$, we deduce

$$\int_0^T \langle \chi - L_{p,\sigma} u, v\rangle\, dt \geqslant 0, \text{ for all } v \in L^p(0, T; \mathcal{D}_0^{1,p}(\Omega, \sigma)).$$

Hence $\chi = L_{p,\sigma} u$. One can check that $\beta(u(0)) = \beta(u_0)$.

ii) Uniqueness. By (3.20), the solutions of (1.1) satisfies the condition

$$\frac{\partial \beta(u)}{\partial t} \in L^2(\tau, T; L^2(\Omega)), \quad \forall \tau > 0.$$

Assume that u, w are weak solutions of problem (1.1) with initial data $\beta(u_0), \beta(v_0)$ in $L^1(\Omega)$. Then, $v := u - w$, $\tilde{v} := \beta(u) - \beta(w)$ satisfy

$$\begin{cases} \tilde{v}_t + (L_{p,\sigma} u - L_{p,\sigma} w) + (f(u) - f(w)) = 0, \quad x \in \Omega, t > 0, \\ v_{|\partial\Omega} = 0, \\ \tilde{v}_{|t=0} = \beta(u_0) - \beta(w_0). \end{cases} \tag{3.37}$$

Multiplying (3.37) by $\text{sign}(v) = \text{sign}(\tilde{v})$, we have

$$\frac{d}{dt}\int_\Omega |\tilde{v}(t)|\, dx + \int_\Omega (L_{p,\sigma}u - L_{p,\sigma}w)\,\text{sign}(v)dx + \int_\Omega (f(u) - f(w))\,\text{sign}(v)dx = 0.$$

If $v \neq 0$ then by Proposition 2.2, we have

$$\int_\Omega (L_{p,\sigma}u - L_{p,\sigma}w)\,\text{sign}(v)dx = \frac{1}{|v|}\langle L_{p,\sigma}u - L_{p,\sigma}w, u - w\rangle \geqslant 0$$

for $u, w \in \mathcal{D}_0^{1,p}(\Omega, \sigma)$. So, we deduce that

$$\frac{d}{dt}\int_\Omega |\tilde{v}(t)|\, dx + \int_\Omega (f(u) - f(w))\,\text{sign}(v)dx \leqslant 0.$$

Integrating from η to t ($0 \leqslant \eta \leqslant t$), we have

$$\int_\Omega |\tilde{v}(t)|dx + \int_\eta^t \int_\Omega (f(u) - f(w))\,\text{sign}(v)dxdt \leqslant \int_\Omega |\tilde{v}(\eta)|dx.$$

Let $\eta \to 0$ and note that $\beta(u_0) - \beta(w_0) = 0$ when $u_0 = w_0$, this equation implies

$$\int_\Omega |\tilde{v}(t)|dx + \int_0^t \int_\Omega (f(u) - f(w))\,\text{sign}(v)dxdt \leqslant 0. \tag{3.38}$$

By applying assumption $(H6)$, equation (3.38) implies that

$$\int_\Omega |\tilde{v}(t)|dx \leqslant C_3 \int_0^t \int_\Omega |\tilde{v}(\tau)|dxd\tau.$$

By the Gronwall inequality, we obtain $\tilde{v}(t) = 0$ or $\beta(u) = \beta(w)$, and then $u = w$.

□

4. Existence of a global attractor

Based on Theorem 3.1, we are able to construct a continuous (nonlinear) semigroup $S(t) : L^2(\Omega) \to L^2(\Omega)$ by setting

$$S(t)u_0 := u(t),$$

here, $u(\cdot)$ is the unique weak solution of (1.1) corresponding to the initial datum u_0. Our objective is to show that $S(t)$ has a global attractor $\mathcal{A}$ in the phase space $L^2(\Omega)$.

Proposition 4.1. *Assume that conditions $(H1) - (H7)$ are satisfied. Then, the semigroup $S(t)$ associated with problem* (1.1) *admits an absorbing set in $\mathcal{D}_0^{1,p}(\Omega, \sigma) \cap L^\infty(\Omega)$.*

Proof. Let u be a solution of (1.1) and let u_ϵ be an approximate solution of (3.1) that converges to u. Fixing a time $0 < \tau \leqslant t$, by applying (3.6) together with Sobolev's injection theorem, we obtain the bound

$$\|u_\epsilon(t)\|_{L^q(\Omega)} \leqslant C_\delta, \quad \text{for any } 1 \leqslant q < \infty, \tag{4.1}$$

the constant C_δ depends on the measure of Ω and the parameter δ, which is defined as

$$\delta = \max(\beta^{-1}(C(\tau)), |\beta^{-1}(-C(\tau))|),$$

as established in (3.6). From estimate (4.1), it follows that

$$\|u(t)\|_{L^q(\Omega)} \leqslant C_\delta, \text{ for any } 1 \leqslant q < \infty. \tag{4.2}$$

Let $q \to +\infty$ in (4.2), we get

$$\|u(t)\|_{L^\infty(\Omega)} \leqslant C_\delta. \tag{4.3}$$

As a result, combining (4.2) and (4.3), we conclude the existence of an absorbing set in $L^q(\Omega)$, $1 \leqslant q \leq \infty$.

Furthermore, from estimate (3.16), we derive the inequality

$$\int_\Omega \sigma(x)|\nabla u(t)|^p dx + \int_\Omega |u(t)|^q dx \leqslant C_{16}(\tau), \text{ for any } t \geqslant \tau.$$

This and (4.3) imply the existence of an absorbing set in $\mathcal{D}_0^{1,p}(\Omega, \sigma) \cap L^\infty(\Omega)$. □

Using Proposition 4.1 together with the compactness of the embedding $\mathcal{D}_0^{1,p}(\Omega, \sigma) \hookrightarrow L^2(\Omega)$, we obtain the following result.

Theorem 4.1. *Suppose that conditions $(H1) - (H7)$ are satisfied. Then, the semigroup $S(t)$ generated by solutions of problem* (1.1) *admits a global attractor $\mathcal{A}$, which is bounded in $\mathcal{D}_0^{1,p}(\Omega, \sigma) \cap L^\infty(\Omega)$, compact and connected in $L^2(\Omega)$. In particular, the domain of attraction of $\mathcal{A}$ is the entire phase space $L^2(\Omega)$.*

REFERENCES

[1] Caldiroli P & Musina R, 2000. On a variational degenerate elliptic problem. *Nonlinear Differential Equations and Applications*, 7, 187-199.

[2] Cung TA, Nguyen DB & Le TT, 2010. On the global attractors for a class of semilinear degenerate parabolic equations. *Annales Polonici Mathematici*, 98, 71-89.

[3] Cung TA, Nguyen DB & Le TT, 2010. Attractors for quasilinear parabolic equationinvolving weighted p-Laplacian operators. *Vietnam Journal of Mathematics*, 38, 261-280.

[4] Cung TA & Tran DK, 2009. Long-time behavior for quasilinear parabolic equations involving weighted p-Laplacian operators. *Nonlinear Analysis*, 71, 4415-4422.

[5] Cung TA & Le TT, 2012. Notes on global attractors for a class of semilinear degenerate parabolic equations. *Journal of Nonlinear Evolution Equations and Applications*, 4, 41-56.

[6] Khanmamedov AK, 2009. Long-time behaviour of doubly nonlinear parabolic equations. *Communications on Pure and Applied Analysis*, 8, 1373-1400.

[7] Miranville A, 2006. Finite dimensional global attractor for a class of doubly nonlinear parabolic equations. *Central European Journal of Mathematics*, 4, 163-182.

[8] Rougirel A, 2008. Convergence to steady state and attractors for doubly nonlinear equations. *Journal of Mathematical Analysis and Applications*, 339, 281-294.

[9] Eden A, Michaux B & Rakotoson JM, 1991. Doubly nonlinear parabolic type equations as dynamical systems. *Differential Equations and Dynamical Systems*, 3, 87-131.

[10] Ladyzenskaya O, Solonnikov VA & Ouraltseva NN, 1968. Linear and Quasilinear Equations of Parabolic Type. *American Mathematical Society*.

[11] Diaz JI & De Thélin F, 1994. On a nonlinear parabolic problem arising in some models related to turbulent flows. *SIAM Journal on Mathematical Analysis SIAM*, 25, 1085-1111.

[12] Temam R, 1997. Infinite Dimensional Dynamical Systems in Mechanics and Physics, 2nd edition. *Springer-Verlag*.

[13] Blanchard D & Francfort GA, 1988. Study of doubly nonlinear heat equation with no growth assumptions on the parabolic term. *SIAM Journal on Mathematical Analysis*, 19, 1032-1056.