HNUE JOURNAL OF SCIENCE
Natural Science, 2025, Volume 70, Issue 2, pp. 23-36
This paper is available online at http://hnuejs.edu.vn/ns
DOI: 10.18173/2354-1059.2025-0018

GLOBAL DISSIPATIVITY OF POSITIVE NEURAL NETWORKS WITH TIME-VARYING DELAYS

Le Nguyen Van Duyen
K71 Student of the Faculty of Mathematics, Hanoi National University of Education, Hanoi city, Vietnam
Corresponding author: Le Nguyen Van Duyen, e-mail: stu715111017@hnue.edu.vn

Received March 27, 2025. Revised May 14, 2025. Accepted June 30, 2025.

Abstract. In this paper, the problems of positivity and dissipativity are investigated for a model of neural networks with time-varying delays. A novel approach based on comparison techniques via differential and integral inequalities is presented and utilized to derive testable conditions which ensure the existence and the global dissipativity of a unique positive equilibrium point. The derived conditions are formulated in terms of linear programming with M-matrix, providing a computationally efficient framework for analysis. A numerical example with simulations is provided to illustrate the theoretical results and demonstrate their practical applicability.
Keywords: positive neural networks, dissipativity, time-varying delay, M-matrix.

1. Introduction

Neural networks models in general and artificial neural networks are found in various areas of practical applications such as time series forecasting, pattern recognition for medical visualization aids, image processing, speech recognition or parallel computation [1]-[4]. In many practical applications using neural networks, it is very important to ensure the existence and stability in some sense of a unique equilibrium [5]. In addition, the implementation of neural networks is often encountered with time delays due to the limit bandwidth or the signal transmission through layers. The presence of time-delay usually makes the system behavior more complicated and unpredictable [6]-[8]. Thus, the problem of long-time behavior analysis of time-delay neural networks models has been extensively studied in the past few decades [9]-[13].

Positive systems are dynamical systems whose states and outputs are always nonnegative subject to nonnegative inputs. This type of system is widely used to describe practical systems with positive constraints on state variables according to the nature of phenomena [14]. Many important problems in the systems and control theory have been extensively studied for positive linear systems with delays. However, this area is still considerably less well-developed for positive nonlinear systems in neural network models. Note also that when an artificial neural network is designed for certain applications of positive systems in identification [15], control [16], monotone-regular behavior implement [17] and other disciplines including computer vision, pattern recognition or alignment and detection [18], state variables are required to inherit positivity constraints. Thus, it is important to study the problem of long-term behavior for positive nonlinear time-delay systems in network structures. In particular, systematic approaches and effective tools used in the analysis of such models are obviously relevant to develop.

In this paper, we consider the problem of dissipativity of a positive neural networks model with multiple time-varying delays. A novel approach based on comparison techniques via differential and integral inequalities will be presented and utilized to derive testable conditions which ensure the existence and the global dissipativity of a unique positive equilibrium point. The derived conditions are formulated in terms of linear programming conditions with matrices, which can be effectively solved by various convex algorithms. A numerical example with simulations is provided to demonstrate the effectiveness of the obtained results.

2. Preliminaries

Notations: $\mathbb{R}^n$ denotes the n-dimensional Euclidean space endowed with the vector norm $\|x\| = \max_{1\leq i\leq n} |x_i|$, $\mathbf{1}_n \in \mathbb{R}^n$ is the vector with all entries equal one. $\mathbb{R}^{m\times n}$ is the set of $m \times n$-matrices. Comparison between vectors $x = (x_i) \in \mathbb{R}^n$ and $y = (y_i) \in \mathbb{R}^n$ is understood componentwise. More precisely, we write $x \preceq y$ if $x_i \leq y_i$ and $x \prec y$ if $x_i < y_i$ for all $i \in [n] \triangleq \{1, 2, \dots, n\}$. $\mathbb{R}^n_+ = \{x \in \mathbb{R}^n : x \succeq 0\}$ and $|x| = (|x_i|) \in \mathbb{R}^n_+$. A matrix $A = (a_{ij}) \in \mathbb{R}^{m\times n}$ is nonnegative, $A \succeq 0$, if $a_{ij} \geq 0$ for all i, j, A is a Metzler matrix if $a_{ij} \geq 0$ for all $i \neq j$. $C([a, b], \mathbb{R}^n)$ denotes the set of $\mathbb{R}^n$-valued continuous functions on $[a, b]$ endowed with the *supremum* norm $\|\phi\|_C = \sup_{a\leq t\leq b} \|\phi(t)\|$.

2.1. Auxiliary results

A vector field $F : \mathbb{R}^n \to \mathbb{R}^n$ is said to be order-preserving on $\mathbb{R}^n_+$ if $F(x) \preceq F(y)$ for any vectors $x \preceq y$ [14]. In addition, a mapping $G : \mathbb{R}^n \to \mathbb{R}^n$ is proper if $G^{-1}(K)$ is a compact set for any compact subset $K \subset \mathbb{R}^n$. It is well-known that a continuous mapping $G : \mathbb{R}^n \to \mathbb{R}^n$ is proper if and only if G has the property that for any sequence $\{p_k\} \subset \mathbb{R}^n$, $\|p_k\| \to \infty$ then $\|G(p_k)\| \to \infty$ as $k \to \infty$.

Lemma 2.1 (see, [19]). *A locally invertible continuous mapping $G : \mathbb{R}^n \to \mathbb{R}^n$ is a homeomorphism of $\mathbb{R}^n$ onto itself if and only if it is proper.*

A matrix $A = (a_{ij}) \in \mathbb{R}^{n\times n}$ is an M-matrix if it can be expressed in the form $A = sI_n - B$, where $B = (b_{ij}) \succeq 0$ and $s \geq \rho(B)$, the maximum of the moduli of eigenvalues of B (also known as the spectral radius of B). An M-matrix A is nonsingular if and only if $s > \rho(B)$. The following proposition summarizes widely used properties of nonsingular M-matrix.

Proposition 2.1 (see, [20]). *Let $A = (a_{ij}) \in \mathbb{R}^{n\times n}$ be an M-matrix. The following statements are equivalent.*

(i) A is a nonsingular M-matrix.

(ii) All the principal minors of A are positive.

(iii) $A + D$ is nonsingular for any nonnegative diagonal matrix D.

(iv) A is inverse-positive, that is, there exists $A^{-1} \succeq 0$.

(v) A has a convergent regular splitting, that is, A has a representation of the form $A = M - N$ with $M^{-1} \succeq 0$, $N \succeq 0$ and $\rho(M^{-1}N) < 1$ ($M^{-1}N$ convergent).

(vi) Every regular splitting of A is convergent.

(vii) There exists a positive vector $p \in \mathbb{R}^n$ such that $Ap \succ 0$.

It follows from Proposition 2.1 that if $P = (p_{ij}) \in \mathbb{R}^{n\times n}$ is a nonnegative matrix whose spectral radius $\rho(P) < 1$ then $(I_n - P)^{-1} \succeq 0$ and there exists a positive vector $\zeta = (\zeta_i)$ such that $(I_n - P)\zeta \succ 0$. Therefore,

$$\sum_{j=1}^{n} p_{ij}\zeta_j < \zeta_i, \quad i \in [n].$$

This fact will be useful for our later derivation.

2.2. Model description and problem formulation

Consider a class of nonlinear time-delay systems given by

$$\begin{aligned} x_j'(t) = & -d_j\beta_j(x_j(t)) + \sum_{k=1}^{N} a_{jk} f_k(x_k(t)) \\ & + \sum_{k=1}^{N} b_{jk} g_k(x_k(t - \tau_k(t))) + w_j, \quad j = 1, 2, \dots, N. \end{aligned} \tag{2.1}$$

System (2.1) is typically used to describe a model of neural networks, called Hopfield neural networks, where N is the number of neurons, $x(t) = (x_j(t)) \in \mathbb{R}^N$ is the state vector whereas $w = (w_j) \in \mathbb{R}^N$ represents the external input vector to the network, $f, g : \mathbb{R}^N \to \mathbb{R}^N$, $f(x) = (f_k(x_k))$ and $g(x) = (g_k(x_k))$ are neuron activation functions, $\beta_k(x_k)$ and $d_k > 0$ are nonlinear self-excitation rates and self-inhibition coefficients, a_{jk} and b_{jk}, $j, k \in [N]$, are neuron connection coefficients, $\tau_k(t)$, $k = 1, 2, \cdots, N$, represent time-varying delays which satisfy $0 \le \tau_k(t) \le \tau^+$ for all $t \ge 0$, where τ^+ is a given constant involving the upper bound of delays. The initial condition of system (2.1) is specified as

$$x(s) = \phi(s), \quad s \in [-\tau^+, 0],$$

where $\phi \in C([-\tau^+, 0], \mathbb{R}^N)$. We define the following set of admissible functions

$$\mathcal{F} = \left\{ \beta \in C(\mathbb{R}, \mathbb{R}) \middle| \beta(0) = 0, \quad r_\beta^- \leq \frac{\beta(u) - \beta(v)}{u - v} \leq r_\beta^+,\ \forall u \neq v \right\},$$

where r_β^-, r_β^+ are some positive scalars. It is clear that the set $\mathcal{F}$ contains all linear functions $\beta(u) = r_\beta u$ with constant $r_\beta > 0$.

Assumption (A): The decay rate functions $\beta_j \in \mathcal{F}$ for all $j \in [N]$ and the activation functions $f_k(.)$ and $g_k(.)$ are continuous which satisfy

$$0 \leq \frac{f_k(u) - f_k(v)}{u - v} \leq l_k^f, \quad 0 \leq \frac{g_k(u) - g_k(v)}{u - v} \leq l_k^g,\ \forall u \neq v,$$

where l_k^f and l_k^g, $k \in [N]$, are positive constants.

Remark 2.1. *By Assumption (A), the functions $f(x) = (f_k(x_k))$ and $g(x) = (g_k(x_k))$ are globally Lipschitz continuous on $\mathbb{R}^n$. Thus, by utilizing fundamental results in the theory of functional differential equations [21], it can be verified that for any initial function $\phi \in C([-\tau^+, 0], \mathbb{R}^N)$, there exists a unique solution $x(t) = x(t, \phi)$ of* (2.1) *on the interval $[0, \infty)$, which is absolutely continuous in t.*

Definition 2.1. *System* (2.1) *is said to be positive if for any initial function $\phi \in C([-\tau^+, 0], \mathbb{R}_+^N)$ and nonnegative input vector $w \in \mathbb{R}_+^N$, the corresponding state trajectory is nonnegative. In other words, system* (2.1) *is positive if*

$$\begin{cases} \phi(s) \succeq 0,\ s \in [-\tau^+, 0], \\ w \succeq 0 \end{cases} \implies x(t) \succeq 0 \text{ for all } t \geq 0.$$

Definition 2.2. *A vector $x_e \in \mathbb{R}^N$ is said to be an equilibrium point (EP) of system* (2.1) *if it satisfies the following equation*

$$-H(x_e) + Af(x_e) + Bg(x_e) = 0, \tag{2.2}$$

where the function $H : \mathbb{R}^N \to \mathbb{R}^N$ is defined as $H(x) = (d_j \beta_j(x_j))$. Moreover, the EP x_e is said to be positive if $x_e \succeq 0$.

Definition 2.3. *A positive EP x_e of system* (2.1) *is said to be globally dissipative if for any compact set $\Omega \subset \mathbb{R}_+^N$, there exists a constant $\eta(\Omega) > 0$ such that*

$$\limsup_{t \to \infty} \|x(t) - x_e\|_\infty \leq \eta(\Omega)$$

for any $w \in \Omega$ and initial function $\phi \in C([-\tau^+, 0], \mathbb{R}_+^N)$.

Our main aim in this paper is to derive tractable conditions by which system (2.1) is positive and possesses a unique positive EP that is globally dissipative.

3. Main results

3.1. Positivity

In this section, we first prove the positivity of system (2.1). That is, for any initial function $\phi \in C([-\tau^+, 0], \mathbb{R}^N_+)$ and input vector $w \in \mathbb{R}^N_+$, the corresponding state trajectory $x(t)$ of the system is nonnegative.

Proposition 3.1. *Let Assumption* (A) *hold. Then, system* (2.1) *is positive for any bounded delays if the connection matrices A and B are nonnegative.*

Proof. Let $x(t)$ be a solution of (2.1) with $\phi(s) \succ 0$ for all $s \in [-\tau^+, 0]$ and $w \in \mathbb{R}^N_+$. We will show that $x(t) \succ 0$ for all $t \geq 0$. By the continuity of $x(t)$, it is assumed in contrast that there are $j \in [N]$ and $t_* > 0$ such that

$$x_j(t_*) = 0, \quad x_j(t) > 0,\ t \in [0, t_*),$$

and $x_k(t) \geq 0$ for all $k \in [N]$. Then,

$$z_j(t) := \sum_{k=1}^{N} a_{jk} f_k(x_k(t)) + \sum_{k=1}^{N} b_{jk} g_k(x_k(t - \tau_k(t))) + w_j \geq 0\ t \in [0, t_*].$$

In addition, by Assumption (A),

$$r_j^- \leq \frac{\alpha_j(x_j(t))}{x_j(t)} \leq r_j^+, \quad t \in [0, t_*).$$

Thus, we have

$$x_j'(t) \geq -r_j^+ x_j(t) + z_j(t),\ t \in [0, t_*). \tag{3.1}$$

Taking integral both sides of inequality (3.1) we then obtain

$$\begin{aligned} x_j(t) &\geq e^{-r_j^+ t}\left(\phi_j(0) + \int_0^t e^{r_j^+ s} z_j(s) ds\right) \\ &\geq e^{-r_j^+ t}\phi_j(0), \quad t \in [0, t_*). \end{aligned} \tag{3.2}$$

Let $t \uparrow t_f$, from (3.2) we get

$$0 < \phi_j(0) e^{-r_j^+ t_*} \leq x_j(t_*) = 0.$$

This contradiction indicates that $x(t) \succ 0$ for $t \in [0, \infty)$.

Now, for a given $\epsilon > 0$, let $x_\epsilon(t)$ be the solution of system (2.1) with initial function $\phi_\epsilon = \phi + \epsilon \mathbf{1}_n$, where $\phi \in C([-\tau^+, 0], \mathbb{R}^N_+)$. By the above arguments, we have $x_\epsilon(t) \succ 0$ for all $t \geq 0$. Thus,

$$x(t) = \lim_{\epsilon \to 0^+} x_\epsilon(t) \succeq 0.$$

The proof is completed. □

3.2. Equilibrium point

It can be verified by equation (2.2) that an EP x_e of system (2.1) exists if and only if the mapping $\Phi : \mathbb{R}^N \to \mathbb{R}^N$ defined by $\Phi(x) = -H(x) + Af(x) + Bg(x)$ has at least a null point. By utilizing Lemma 2.1, we have the following result.

Theorem 3.1. *Let assumption* (A) *hold. Assume that the connection matrices A and B are nonnegative and there exists a positive vector $\xi \in \mathbb{R}^N$, $\xi \succ 0$, that satisfies the following condition*

$$\sum_{j=1}^{N} \left(a_{jk} l_k^f + b_{jk} l_k^g \right) \xi_j < d_k r_{\beta_k}^- \xi_k, \; k \in [N], \tag{3.3}$$

or in vector form

$$\xi^\top \left(-DR_\beta^- + AL_f + BL_g \right) \prec 0,$$

where $D = \{d_1, \ldots, d_N\}$, $R_\beta^- = \{r_{\beta_1}^-, \ldots, r_{\beta_N}^-\}$, $L_f = \{l_1^f, \ldots, l_N^f\}$ and $L_g = \{l_1^g, \ldots, l_N^g\}$. Then, system (2.1) *has a unique EP $x_e \in \mathbb{R}^N$.*

Proof. For any $u, v \in \mathbb{R}^N$, we have

$$\Phi(u) - \Phi(v) = -(H(u) - H(v)) + A[f(u) - f(v)] + B[g(u) - g(v)]. \tag{3.4}$$

By Assumption (A), we have

$$\begin{aligned} (u_j - v_j)\left(\beta_j(u_j) - \beta_j(v_j)\right) &\geq r_{\beta_j}^- |u_j - v_j|, \\ (u_k - v_k)(f_k(u_k) - f_k(v_k)) &\leq l_k^f |u_k - v_k|. \end{aligned}$$

By multiplying both sides of (3.4) with $S(u - v) \triangleq \{(u_j - v_j)\}$, we obtain

$$S(u - v)\left(\Phi(u) - \Phi(v)\right) \preceq \left(-DR_\beta^- + AL_f + BL_g \right) |u - v|.$$

Consequently, we have

$$|\Phi(u) - \Phi(v)| \succeq \left(DR_\beta^- - AL_f - BL_g \right) |u - v|. \tag{3.5}$$

For a vector $\xi \in \mathbb{R}^N$, $\xi \succ 0$, it follows from (3.5) that

$$\xi^\top |\Phi(u) - \Phi(v)| \succeq \xi^\top \left(DR_\beta^- - AL_f - BL_g \right) |u - v|.$$

Therefore, if $\Phi(u) = \Phi(v)$ then it follows from condition (3.3) that

$$\xi^\top \left(DR_\beta^- - AL_f - BL_g \right) |u - v| = 0$$

and thus $|u - v| = 0$. This clearly yields $u = v$ and therefore the mapping Φ is an injective continuous mapping.

On the other hand, the inequality (3.5) also gives

$$\|\Phi(u)\| \geq \frac{1}{\|\xi\|}\xi^\top \left(DR_\beta^- - AL_f - BL_g\right)|u| - \|\Phi(0)\|.$$

The last inequality guarantees that $\|\Phi(u_k)\| \to \infty$ for any sequence $\{u_k\} \subset \mathbb{R}^N$ with $\|u_k\| \to \infty$. By Lemma 2.1, the mapping Φ is a homeomorphism onto $\mathbb{R}^N$. Therefore, the equation $\Phi(x) = 0$ has a unique solution $x_e \in \mathbb{R}^N$ which is an EP of (2.1). The proof is completed. □

Remark 3.1. *The proof of Theorem 3.1 based on the properties of homeomorphisms does not guarantee the positivity of the EP x_e. This property will be shown as a consequence of the global dissipativity of the EP x_e.*

3.3. Global dissipativity of positive EP

In this section, we will prove that, under the derived conditions in Theorem 3.1, the EP x_e is positive and globally dissipative. It is noticed that since the matrices $\Xi = -DR_\beta^- + AL_f + BL_g$ and $\Xi^\top$ are both Metzler matrices and condition (3.3) is equivalent to $\Xi^\top\xi < 0$. This condition is feasible for a vector $\xi \succ 0$ if and only if $\Xi^\top$ and Ξ are Metzler-Hurwitz matrices [20]. In the following, we will show that the derived conditions in Proposition 3.1 and 3.1 ensure that the system (2.1) is positive and the unique EP x_e is positive and globally dissipative.

Theorem 3.2. *Let Assumption* (A) *hold. Assume that the connection matrices A and B are nonnegative and there exists a positive vector $\hat{\xi} \in \mathbb{R}^N$, $\hat{\xi} \succ 0$, such that*

$$\Xi\hat{\xi} = \left(-DR_\beta^- + AL_f + BL_g\right)\hat{\xi} \prec 0. \tag{3.6}$$

Then, system (2.1) *has a unique positive EP $x_e \in \mathbb{R}_+^N$ which is globally dissipative. Moreover, there exists a positive constant λ such that, for any compact set $\Omega \subset \mathbb{R}_+^N$ defined as a box*

$$\Omega = \left\{\eta \in \mathbb{R}^N : 0 \preceq \eta \preceq \eta^*\right\},$$

where η^* is a given positive vector, it holds that

$$\limsup_{t\to\infty} \|x(t) - x_e\| \leq \lambda\|\eta^*\|$$

for any solution $x(t)$ of (2.1) *with initial function $\phi \in C([-\tau^+, 0], \mathbb{R}_+^N)$ and input vector $w \in \Omega$.*

Proof. By Proposition 3.1, system (2.1) is positive. In addition, according to Theorem 3.1, there exists a unique EP $x_e \in \mathbb{R}^N$ of (2.1).

We first prove the dissipativity of the EP $x_e = (x_j^e) \in \mathbb{R}^N$. Indeed, let $x(t) = (x_j(t))$ be a solution of (2.1) with initial function $\phi \in C([-\tau^+, 0], \mathbb{R}_+^N)$ and input vector $w \in \Omega$. Then, we have

$$\begin{aligned}\big(x_j(t) - x_j^e\big)' = & - d_j \left(\beta_j(x_j(t)) - \beta_j(x_j^e)\right) + \sum_{k=1}^N a_{jk}[f_k(x_k(t)) - f_k(x_k^e)] \\ & + \sum_{k=1}^N b_{jk}[g_k(x_k(t-\tau_k(t))) - g_k(x_k^e)] + w_j. \end{aligned} \tag{3.7}$$

We define the vector-valued function $v(t) = |x(t) - x_e| = (v_j(t))$, $t \geq -\tau^+$. Then, it follows from (3.7) that

$$\begin{aligned} D^+ v_j(t) &= (x_j(t) - x_j^e)(x_j(t) - x_j^e)' \\ &\leq -d_j r_{\beta_j}^- |x_j(t) - x_j^e| + \sum_{k=1}^N a_{jk} l_k^f |x_k(t) - x_k^e| \\ &\quad + \sum_{k=1}^N b_{jk} l_k^g |x_k(t-\tau_k(t)) - x_k^e| + w_j,\ t \geq 0. \end{aligned} \tag{3.8}$$

where $D^+ v_j(t)$ denotes the upper-left Dini derivative of $v_j(t)$. From (3.8) we obtain

$$D^+ v_j(t) \leq -d_j r_{\beta_j}^- v_j(t) + \sum_{k=1}^N a_{jk} l_k^f v_k(t) + \sum_{k=1}^N b_{jk} l_k^g v_k(t-\tau_k(t)) + \eta_j^*. \tag{3.9}$$

We will establish an exponential convergence of $v(t)$ within a specified threshold. For this, observe from (3.6) that

$$-d_j r_{\beta_j}^- \hat{\xi}_j + \sum_{k=1}^N (a_{jk} l_k^f + b_{jk} l_k^g) \hat{\xi}_k < 0, \quad j \in [N]. \tag{3.10}$$

Moreover, in (3.10), we can replace $\hat{\xi}_j$ by $\tilde{\xi}_j = \hat{\xi}_j / \|\hat{\xi}\|$ then $\|\tilde{\xi}\| = 1$. Thus, without loss of generality, we can assume that $\|\hat{\xi}\| = 1$. We define the following constants

$$\begin{aligned} &\epsilon_* = \min_{j\in[N]} \left\{ d_j r_{\beta_j}^- \hat{\xi}_j - \sum_{k=1}^N (a_{jk} l_k^f + b_{jk} l_k^g) \hat{\xi}_k \right\} > 0, \\ &\delta_* = \frac{\epsilon_*}{\hat{\xi}_+}, \quad \hat{\xi}_+ = \min_{1\leq j\leq N} \hat{\xi}_j, \quad R_1 = \frac{\|\eta^*\|}{\delta_*}, \quad R_2 = \frac{\|\eta^*\|}{\epsilon_*}. \end{aligned}$$

Since $\hat{\xi}_+ \leq \|\hat{\xi}\| = 1$, we have $R_1 \leq R_2$. In addition,

$$-d_j r_{\beta_j}^- \hat{\xi}_j + \sum_{k=1}^N (a_{jk} l_k^f + b_{jk} l_k^g) \hat{\xi}_k \leq -\epsilon_*$$

for all $j \in [N]$.

We first show that if $\|\phi - x_e\|_C \leq R_1$ then $\|v(t)\| \leq R_2$ for all $t \geq 0$. Specifically, let $\|\phi - x_e\|_C \leq R_1$ then

$$v_j(t) \leq \sup_{-\tau^+ \leq t \leq 0} |\phi_j(t) - x_j^e| \leq R_1 \leq \hat{\xi}_j R_2, \quad t \in [-\tau^+, 0].$$

If there exist an index $j \in [N]$ and a $\hat{t} > 0$ such that

$$v_j(\hat{t}) = \hat{\xi}_j R_2, \quad v_k(t) \leq \hat{\xi}_k R_2, \; t \in [0, \hat{t}]$$

then, from (3.9), we have

$$\begin{aligned} D^+ v_j(\hat{t}) &\leq R_2 \Big(-d_j r_{\beta_j}^- \hat{\xi}_j + \sum_{k=1}^N (a_{jk} l_k^f + b_{jk} l_k^g) \hat{\xi}_k \Big) + \eta_j^* \\ &\leq -\epsilon_* R_2 + \eta_j^* \leq 0. \end{aligned}$$

Thus, $v_j(t)$ cannot exceed $\hat{\xi}_j R_2$ for all $t \geq 0$. This shows that

$$\|v(t)\| \leq R_2 \|\hat{\xi}\| = R_2, \quad t \geq 0.$$

Assume that $\|\phi - x_e\|_C > R_1$. Then, there exists a positive scalar β such that

$$\|\phi_j - x_j^e\|_C - R_2 \hat{\xi}_j < \beta \left(\|\phi - x_e\|_C - R_1 \right) \hat{\xi}_j, \quad j \in [N].$$

Let us define the functions $\Delta_j : [0, \infty) \to \mathbb{R}$, $j \in [N]$, as

$$\Delta_j(\theta) = \theta \hat{\xi}_j + \Big(\sum_{k=1}^N b_{jk} l_k^g \hat{\xi}_k \Big) \Big(e^{\theta \tau^+} - 1 \Big) - \epsilon_*.$$

Clearly, for each $j \in [N]$, $\Delta_j(\theta)$ is a continuous function on $[0, \infty)$, $\Delta_j(0) < 0$ and $\Delta_j(\theta) \to \infty$ as $\theta \to \infty$. Thus, there exists a unique positive solution θ_j^* of the scalar equation $\Delta_j(\theta) = 0$. Let $\sigma = \min_{1 \leq j \leq N} \theta_j^*$. Since the function $\Delta_j(\theta)$ is increasing in θ, we have

$$\Delta_j(\sigma) \leq \Delta_j(\theta_j^*) = 0$$

and hence $\Delta_j(\sigma) \leq 0$ for all $j \in [N]$.

Consider the following functions

$$\rho_j(t) = \underbrace{\beta \left(\|\phi - x_e\|_C - R_1 \right)}_{\kappa} \hat{\xi}_j e^{-\sigma t}, \quad t \geq 0, \; j \in [N].$$

and $\rho_j(t) = \rho_j(0)$ for $t \in [-\tau^+, 0]$. Note also that for any $k \in [N]$, we have

$$\begin{aligned} \rho_k(t - \tau_k(t)) &= \kappa \hat{\xi}_k e^{-\sigma(t - \tau_k(t))} \\ &= e^{\sigma \tau_k(t)} \rho_k(t) \leq e^{\sigma \tau^+} \rho_k(t), \quad t \geq 0. \end{aligned}$$

Therefore,

$$
\begin{aligned}
-d_j r_{\beta_j}^- \rho_j(t) &+ \sum_{k=1}^N a_{jk} l_k^f \rho_k(t) + \sum_{k=1}^N b_{jk} l_k^g \rho_k(t-\tau_k(t)) \\
&\le \Big[- d_j r_{\beta_j}^- \hat{\xi}_j + \sum_{k=1}^N a_{jk} l_k^f \hat{\xi}_k + \Big(\sum_{k=1}^N b_{jk} l_k^g \hat{\xi}_k \Big) e^{\sigma\tau^+} \Big] \kappa e^{-\sigma t} \\
&\le \Big[- \epsilon_* + \Big(\sum_{k=1}^N b_{jk} l_k^g \hat{\xi}_k \Big) \Big(e^{\sigma\tau^+} - 1 \Big) \Big] \kappa e^{-\sigma t} \\
&\le (\Delta_j(\sigma) - \sigma\hat{\xi}_j) \kappa e^{-\sigma t}.
\end{aligned} \tag{3.11}
$$

It can be deduced from (3.11) that

$$
\rho_j'(t) \ge -d_j r_{\beta_j}^- \rho_j(t) + \sum_{k=1}^N a_{jk} l_k^f \rho_k(t) + \sum_{k=1}^N b_{jk} l_k^g \rho_k(t-\tau_k(t)), \quad t \ge 0. \tag{3.12}
$$

We now define the following transformation

$$
\varphi_j(t) = v_j(t) - R_2 \hat{\xi}_j, \quad j \in [N]. \tag{3.13}
$$

It follows from (3.9) that

$$
\begin{aligned}
D^+\varphi_j(t) \le \; & - d_j r_{\beta_j}^- \varphi_j(t) + \sum_{k=1}^N a_{jk} l_k^f \varphi_k(t) + \sum_{k=1}^N b_{jk} l_k^g \varphi_k(t-\tau_k(t)) \\
& + R_2 \Big(- d_j r_{\beta_j}^- \hat{\xi}_j + \sum_{k=1}^N (a_{jk} l_k^f + b_{jk} l_k^g) \hat{\xi}_k \Big) + \eta_j^*.
\end{aligned} \tag{3.14}
$$

Since

$$
R_2 \Big(- d_j r_{\beta_j}^- \hat{\xi}_j + \sum_{k=1}^N (a_{jk} l_k^f + b_{jk} l_k^g) \hat{\xi}_k \Big) + \eta_j^* \le -R_2 \epsilon_* + \eta_j^* \le 0,
$$

from (3.14), we obtain

$$
D^+\varphi_j(t) \le -d_j r_{\beta_j}^- \varphi_j(t) + \sum_{k=1}^N a_{jk} l_k^f \varphi_k(t) + \sum_{k=1}^N b_{jk} l_k^g \varphi_k(t-\tau_k(t)), \quad t \ge 0. \tag{3.15}
$$

Finally, we will prove that $\varphi_j(t) \le \rho_j(t)$. Let $\chi_j(t) = \varphi_j(t) - \rho_j(t)$. It follows from (3.12) and (3.15) that

$$
D^+\chi_j(t) \le -d_j r_{\beta_j}^- \chi_j(t) + \sum_{k=1}^N a_{jk} l_k^f \chi_k(t) + \sum_{k=1}^N b_{jk} l_k^g \chi_k(t-\tau_k(t)), \quad t \ge 0. \tag{3.16}
$$

For $t \in [-\tau^+, 0]$, we have

$$\begin{aligned}\varphi_j(t) &\le \|\phi_j - x_j^e\|_C - R_2\hat{\xi}_j \\ &< \beta(\|\phi\| - x_e\|_C - R_1)\hat{\xi}_j = \rho_j(0).\end{aligned}$$

Thus, $\chi_j(t) < 0$ for $t \in [-\tau^+, 0]$. Assume that there exist an index $j \in [N]$ and a $t_1 > 0$ such that

$$\chi_j(t_1) = 0,\ \chi_j(t) < 0,\ t \in [-\tau^+, t_1),\ \text{and}\ \chi_k(t) \le 0,\ t \in [-\tau^+, t_1].$$

From (3.16), we have

$$D^+\chi_j(t) \le -d_j r_{\beta_j}^- \chi_j(t), \quad t \in [0, t_1),$$

which gives

$$\chi_j(t) \le \chi_j(0)e^{-d_j r_{\beta_j}^- t}, \quad t \in [0, t_1). \tag{3.17}$$

Let $t \uparrow t_1$, ti can be deduced from (3.17) that

$$0 = \chi_j(t_1) \le \chi_j(0)e^{-d_j r_{\beta_j}^- t_1} < 0.$$

This clearly raises a contradiction. Therefore,

$$\begin{aligned}v_j(t) &= \varphi_j(t) + R_2\hat{\xi}_j \le R_2\hat{\xi}_j + \rho_j(t) \\ &\le \hat{\xi}_j\left(R_2 + \beta(\|\phi - x_e\|_C - R_1)e^{-\sigma t}\right)\end{aligned}$$

and, consequently, we can get

$$\|x(t) - x_e\| = \|v(t)\| \le R_2 + \beta(\|\phi - x_e\|_C - R_1)e^{-\sigma t}$$

by which we readily obtain

$$\limsup_{t\to\infty} \|x(t) - x_e\| \le \frac{\|\eta^*\|}{\epsilon_*}.$$

Finally, let $\tilde{x}(t)$ be the solution of (2.1) with initial function $\phi \in C([-\tau^+, 0], \mathbb{R}^N)$, $\phi(s) \succ 0$, $s \in [-\tau^+, 0]$, and input $w = 0$. By Proposition 3.1, the corresponding trajectory $\tilde{x}(t) \succeq 0$ for all $t \ge 0$. Moreover, we select η^* as $\gamma\epsilon_*\mathbf{1}_N$ for $\gamma > 0$ then

$$0 \le \limsup_{t\to\infty} \|\tilde{x}(t) - x_e\| \le \gamma$$

for any $\gamma > 0$. Let $\gamma \to 0^+$ we then obtain $x_e = \lim_{t\to\infty} \tilde{x}(t) \succeq 0$. This shows that x_e is the unique positive EP of system (2.1). The proof is completed. □

4. Simulations

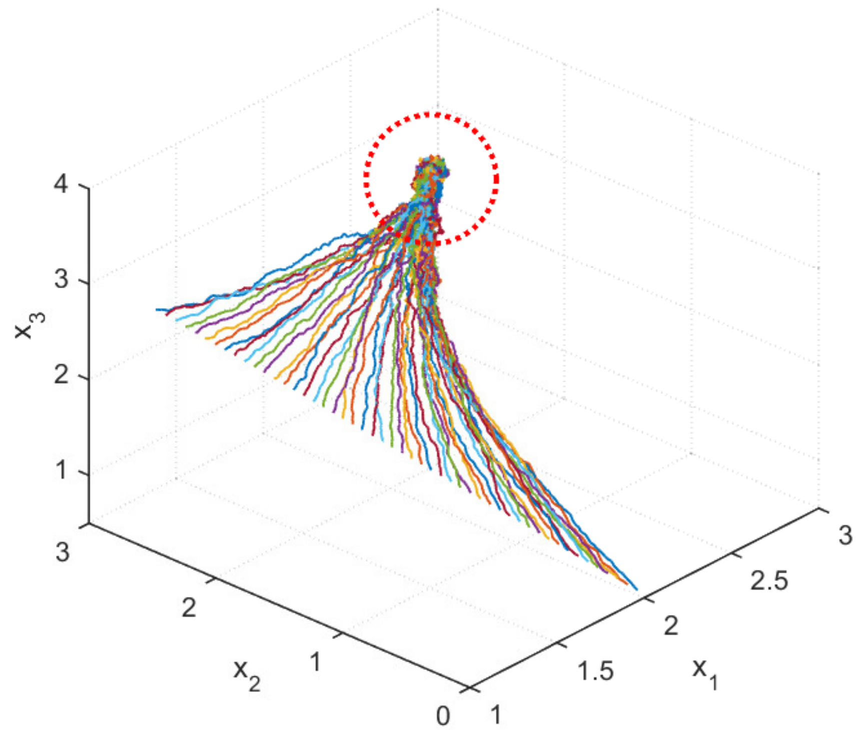

Figure 1. Convergence within the ball $B(x_e, 1/\epsilon_*)$

Consider system (2.1) with linear decay rates $\beta_j(x_j) = x_j$ and Bolzmann sigmoid-type activation functions

$$f_j(x_j) = g_j(x_j) = \nabla_j + \frac{1 - \exp(-\frac{x_j}{\vartheta_j})}{1 + \exp(-\frac{x_j}{\vartheta_j})}, \quad \nabla_j > 0, \quad \vartheta_j > 0. \tag{4.1}$$

The decay rates and connection matrices are given by

$$D = \{1.0, 1.2, 1.5\}$$

and

$$A = \begin{bmatrix} 0.5 & 0.6 & 0.25 \\ 0.7 & 0.4 & 0 \\ 0.2 & 0.15 & 0.1 \end{bmatrix}, \quad B = \begin{bmatrix} 0.15 & 0 & 0.35 \\ 0.2 & 0.3 & 0.3 \\ 0.1 & 0.4 & 0.25 \end{bmatrix}.$$

It is clearly that Assumption (A) is satisfied with $l_j^f = l_j^g = \frac{1}{2\vartheta_j}$. Let $\vartheta_1 = 2, \vartheta_2 = 1$, $\vartheta_3 = 1$ and $\nabla_1 = 0.5, \nabla_2 = 0.8, \nabla_3 = 1$. We have $L_f = L_g = \{0.25, 0.5, 0.5\}$ and hence

$$(-D + AL_f + BL_g)\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -0.2375 \\ -0.475 \\ -0.975 \end{bmatrix} \preceq -\epsilon_* \mathbf{1}_3,$$

where $\epsilon_* = 0.2375$. By Theorem 3.1 system (2.1) has a unique positive EP $x_e \in \mathbb{R}^3_+$. By solving equation (2.2) using Matlab Toolbox, the EP x_e is obtained as

$$x_e = (2.0666, 2.2391, 1.3265)^\top.$$

For illustrative purpose, we fix $\eta^* = \mathbf{1}_3$ then we have

$$\limsup_{t\to\infty} \|x(t) - x_e\| \leq \frac{1}{\epsilon_*} = 4.2105.$$

A phase diagram of 50 state trajectories of the system with delay $\tau_j(t) = 5|\sin(0.2t)|$ is presented in Figure **??**. It can be seen that the conducted state trajectories converge within a ball centered at the positive EP x_e. This validates the obtained theoretical results.

5. Conclusions

This paper investigated the dissipativity of a positive neural networks model with time-varying delays. Based on some novel comparison techniques via differential and integral inequalities, sufficient conditions have been formulated to ensure the existence and global dissipativity of a unique positive EP of the system. A numerical example with simulation have been presented to demonstrate the effectiveness of the derived theoretical results.

REFERENCES

[1] Souli F & Gallinari P, (1998). *Industrial applications of neural networks*. World Scientific Publishing, Singapore.

[2] Venketesh P & Venkatesan R, (2009). A survey on applications of neural networks and evolutionary techniques in web caching. *IETE Technical Review*, 26(3), 171-180.

[3] Raschman E, Zlusk R & Durakov D, (2010). New digital architecture of CNN for pattern recognition. *Journal of Electrical Engineering*, 61(4), 222-228.

[4] Huang G, Huang GB, Song S & You K, (2015). Trends in extreme learning machines: A review. *Neural Networks*, 61, 32-48.

[5] Zhang H, Wang Z & Liu D, (2014). A comprehensive review of stability analysis of continuous-time recurrent neural networks. *IEEE Transactions on Neural Networks and Learning Systems*, 25(7), 1229-1262.

[6] Baldi P & Atiya AF, (1995). How delays affect neural dynamics and learning. *IEEE Transactions on Neural Networks*, 5(5), 612-621.

[7] Lu H, (2012), Chaotic attractors in delayed neural networks, *Physics Letters A* 298(2-3), 109-116.

[8] Park JH, Lee TH, Liu Y & Chen J, (2019). *Dynamic systems with time delays: Stability and control*, Springer-Nature, Singapore.

[9] Le VH & Doan TS, (2015). Finite-time stability of a class of non-autonomous neural networks with heterogeneous proportional delays. *Applied Mathematics and Computation*, 251, 14-23.

[10] Arik S, (2016). Dynamical analysis of uncertain neural networks with multiple time delays. *International Journal of Systems Science*, 47(4), 730-739.

[11] Lee TH, Trinh H & Park JH, (2018). Stability analysis of neural networks with time-varying delay by constructing novel Lyapunov functionals. *IEEE Transactions on Neural Networks and Learning Systems*, 29(9), 4238-4247.

[12] Ge C, Park JH, Hua C & Shi C, (2019). Robust passivity analysis for uncertain neural networks with discrete and distributed time-varying delays. *Neurocomputing*, 364, 330-337.

[13] He J, Liang Y, Yang F & Yang F, (2020). New $\mathcal{H}_\infty$ state estimation criteria of delayed static neural networks via the Lyapunov-Krasovskii functional with negative definite terms. *Neural Networks,* 123, 236-247.

[14] Smith H, (2008). *Monotone Dynamical Systems: An Introduction to the Theory of Competitive and Cooperative Systems*. American Mathematical Society, Providence.

[15] Mzaryn J & Kurek JE, (2010). Design of a neural network for an identification of a robot model with a positive definite inertia matrix, In: *Artificial Intelligence and Soft Computing*, Springer-Verlag, Berlin.

[16] Ma GJ, Wu S & Cai GQ, (2013). Neural networks control of the Ni-MH power battery positive mill thickness. *Applied Mechanics and Materials*, 411-414, 1855-1858.

[17] Ameloot TJ & Van den Bussche J, (2015). Positive neural networks in discrete time implement monotone-regular behaviors. *Neural Computation*, 27(12), 2623-2660.

[18] Abiodun OI, Jantan A, Omolara AE, Dada KV, Mohamed NA & Arshad H, (2018). State-of-the-art in artificial neural network applications: A survey, *Heliyon*, 4, e00938.

[19] Forti M & Tesi A, (1995). New conditions for global stability of neural networks with application to linear and quadratic programming problems. *IEEE Transactions on Circuits and Systems I: Fundamental Theory and Applications*, 42(5), 354-366.

[20] Berman A & Plemmons RJ, (1984). *Nonnegative matrices in the mathematical sciences*. Society for Industrial and Applied Mathematics, Philadelphia.

[21] Arino O, Hbid ML & Ait Dads E, (2002). *Delay differential equations and applications*. Springer, Dordrecht.