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GLOBAL ATTRACTORS TO THREE-DIMENSIONAL NAVIER-STOKES EQUATIONS WITH DAMPING AND FINITE DELAYS

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Abstract. The asymptotic behavior of solutions to the three-dimensional NavierStokes equations is a longstanding and challenging open problem, which has seen numerous important contributions in recent years. In this paper, we consider the NavierStokes equations with damping and delay terms on a bounded domain $\Omega \subset \mathbb{R}^3$. Using the energy method, we prove the existence of global attractors, thus establishing the long-time stability and compactness of the solution set for this class of delayed fluid dynamic models.
Keywords: Navier-Stokes equation, global attractors, delays, damping.

1. Introduction

Let Ω be a bounded domain in $\mathbb{R}^3$, with smooth boundary $\partial\Omega$. We consider the following three-dimensional Navier-Stokes equations with damping and delays

$$\begin{cases} \partial_t u - \nu\Delta u + (u\cdot\nabla)u + \nabla p + \alpha|u|^{\beta-1}u & = h(x) + G(u(t-\rho(t))), \\ & \qquad\qquad (t,x) \in (0,T)\times\Omega, \\ \mathrm{div} u & = 0 \qquad (t,x) \in (0,T)\times\Omega, \\ u(x,t) & = 0 \qquad (t,x) \in (0,T)\times\Gamma, \\ u(0,x) & = u_0(x), \qquad x \in \Omega, \\ u(t,x) & = \phi(t,x), \quad (t,x) \in (-r,0)\times\Omega, \end{cases} \tag{1.1}$$

where $\nu > 0$ is the kinematic viscosity, $\alpha > 0$ and $\beta \geq 1$ are two constants, $u = u(x,t) = (u_1, u_2, u_3)$ is the velocity field of the fluid, p is the pressure, h is a nondelayed external force field, G is another external force term. G incorporates memory effects over a fixed time interval of length $r > 0$, ρ is a given delay function, u_0 represents the initial velocity, and ϕ is the initial datum over the interval.

The equations (1.1) in the case $\alpha \equiv 0$ and $G \equiv 0$ corresponds to the classical Navier-Stokes problem. In [1], the authors studied the case of $G \equiv 0$, that is three-dimensional Navier-Stokes equations involved damping. A number of damping results from resistance to the motion of the flow and applications to various physical scenarios, including porous media flow, drag or friction effects, and certain dissipative mechanisms, have been established (see [2]-[5]).

Navier-Stokes equations with delays in the case of two dimension have been studied by Caraballo T & et al. [6]. In [7], the authors studied three-dimensional Navier-Stokes equations of forcing term with bounded variable delays. It is assumed that $G : \mathbb{R}^3 \to \mathbb{R}^3$ is a measurable function satisfying $G(0) = 0$, and there exists $L_G > 0$ such that

$$|G(u) - G(v)|_{\mathbb{R}^3} \leq L_G |u - v|_{\mathbb{R}^3}, \tag{1.2}$$

for all $u, v \in \mathbb{R}^3$. Let $\rho(t) \in C^1([0,T])$, $\rho(t) \geq 0$ for all $t \in [0,T]$, $r = \max_{t\in[0,T]} \rho(t) > 0$ and $\rho_* = \max_{t\in[0,T]} \rho'(t) < 1$. Galerkin method is utilized to show the existence of weak solutions to (1.1) (see [6], [8]). To establish a priori estimates for $u(t - \rho(t))$, the technique of changing variables proposed in [9] was taken in use.

In this work, we revisit to study equation (1.1) and consider the existence of global attractors to this model. We will use the energy method to show the existence of global attractors. The remainder of the paper is organized as follows: In Section 2, we introduce some function spaces and lemmas that will be utilized in this study. Section 3 focuses on establishing the existence of global attractors for the model.

2. Preliminaries

We consider the following functional space:

$$\mathcal{V} = \big\{ u \in (C_0^\infty(\Omega))^3 \,:\, \nabla \cdot u = 0 \big\}.$$

Let H be the closure of $\mathcal{V}$ in $(L^2(\Omega))^3$ equipped with the norm $|\cdot|$ and inner product $(\cdot,\cdot)$ defined by

$$(u, v) = \sum_{i=1}^{3} \int_\Omega u_i(x) v_i(x) dx,$$

for $u, v \in (L^2(\Omega))^3$. We denote V as the closure of $\mathcal{V}$ in $(H_0^1(\Omega))^3$, equipped with norm $\|\cdot\|$ and associated scalar product $((\cdot,\cdot))$ defined by

$$((u, v)) = \sum_{i,j=1}^{3} \int_\Omega \frac{\partial u_j}{\partial x_i} \frac{\partial v_j}{\partial x_i} dx \text{ for } u, v \in (H_0^1(\Omega))^3.$$

We use $\|\cdot\|_*$ to denote the norm in V' and $\langle\cdot,\cdot\rangle_{V,V'}$ for the dual pairing between V and V'. The Stokes operator $A : V \to V'$ is defined by $\langle Au, v\rangle = ((u, v))$. Let P be the Helmholtz-Leray orthogonal projection in $(H_0^1(\Omega))^3$ onto the space V. Then, for all

$u \in D(A) = (H^2(\Omega))^3 \cap V$, we have $Au = -P\Delta u$. We know that there exists a complete orthonormal set of eigenfunctions $\{w_j\}_{j=1}^{\infty} \subset H$ such that $Aw_j = \lambda_j w_j$ and

$$0 < \lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \cdots \leq \lambda_j \to +\infty \text{ as } j \to \infty.$$

The following inequalities are typically referred as Poincaré inequalities

$$\|u\|^2 \geq \lambda_1 |u|^2, \quad \text{for all } u \in V, \tag{2.1}$$

$$|u|^2 \geq \lambda_1 \|u\|_*^2, \quad \text{for all } u \in H.$$

We define the trilinear form b on $V \times V \times V$ by

$$b(u, v, w) = \sum_{i,j=1}^{3} \int_{\Omega} u_i \frac{\partial v_j}{\partial x_i} w_j dx,$$

for all $u, v, w \in V$, and $B : V \times V \to V'$ by $\langle B(u, v), w\rangle = b(u, v, w)$. We can express $B(u, v) = P[(u \cdot \nabla)v]$. It is easily verified that for all $u, v, w \in V$, the trilinear form satisfies $b(u, v, w) = -b(u, w, v)$,. In particular,

$$b(u, v, v) = 0,$$

for all $u, v \in V$. By using the Hölder and Ladyzhenskaya inequalities, we can determine an optimal positive constant c_0 such that

$$|b(u, v, w)| \leq c_0 \|u\| \|v\| |w|^{1/2} \|w\|^{1/2}, \tag{2.2}$$

for all $u, v, w \in V$. From (2.2) and using Poincaré's inequalities (2.1), we obtain that

$$|b(u, v, w)| \leq c_0 \lambda_1^{-1/4} \|u\| \|v\| \|w\|,$$

for all $u, v, w \in V$. We recall the following lemma from [10].

$$|b(u, v, u)| \leq c_1 |u| \|u\| \|v\|, \tag{2.3}$$

for all $u, v \in V$.

In [7], we presented the following definition of a weak solution.

Definition 2.1. *A function*

$$u \in L^2_{loc}([-r, T]; V) \cap C([-r, T]; H) \cap L^{\beta+1}_{loc}([-r, T]; L^{\beta+1}(\Omega))$$

is said to be a weak solution of (1.1) *if for all* $T > 0$*,*

$$u \in L^{\infty}(0, T; H) \cap L^2(-r, T; V)$$

such that, for all $v \in V$*,*

$$\begin{aligned}\frac{d}{dt}(u(t),v)+\nu((u(t),v))+&b(u(t),u(t),v)+\alpha(|u|^{\beta-1}u,v)\\ &=(h,v)+(G(u(t-\rho(t))),v),\\ u(0)&=u_0,\\ u(t)&=\phi(t),\quad t\in(-r,0).\end{aligned}$$

We also established the existence and uniqueness of weak solutions (see [7, Theorem 3.1]):

Theorem 2.1. *Let assumption* (1.2) *hold and assume* $u_0, h \in H$ *and* $\phi \in L^2(-r,0;H)$*. Then, a unique solution to* (1.1) *exists if* $\nu^2 > \dfrac{L_G}{\lambda_1^2(1-\rho_*)}$.

3. Existence of a global attractor

The initial condition u_s defined on $[-r,0]$ to H is given by

$$u_s(\theta)=u(s+\theta)\text{ for all }\theta\in[-r,0],\ \text{for any }s\in[0,T].$$

If $\rho(t)=-\theta$, $\theta\in[-r,0]$ then $u_t(\theta)=u(t+\theta)=u(t-\rho(t))$. We define a mapping $S(t): L^2(-r,0;H)\times H\to L^2(-r,0;H)\times H$ satisfying

$$S(t)(\phi,u_0)=(u_t,u(t)),$$

where $u(t)$ is a unique solution of (1.1) with initial history ϕ and initial state u_0. The family $\{S(t)\}_{t\geq 0}$ satisfies the semigroup property, that is, $S(0)=I$ and

$$S(t+s)=S(t)\circ S(s),\quad \forall\, t,s\geq 0.$$

This property follows from the causality and well-posedness of the delayed system. Specifically, by solving the system on the interval $[0,t+s]$ from the initial data (ϕ,u_0) yields the same result as solving on $[0,s]$ to obtain $(u_s,u(s))$, and then using this as new initial data to solve from time s to $s+t$. Hence, the operator composition satisfies

$$S(t+s)(\phi,u_0)=S(t)(u_s,u(s))=S(t)\circ S(s)(\phi,u_0).$$

The semigroup property is fundamental for analyzing the long-time behavior of the system, particularly in the construction of a global attractor. It ensures that the evolution defines a well-posed dynamical system, to which the classical theory of infinite-dimensional dissipative systems can be utilizable.

We have the following continuity results.

Lemma 3.1. *Let* (1.2) *hold and assume* $u_0,h\in H$ *and* $\phi\in L^2(-r,0;H)$*. Then, the mapping* $S(t): L^2(-r,0;H)\times H\to L^2(-r,0;H)\times H$ *is continuous for any* $t\in[0,T]$.

Proof. Let $(\phi, u_0), (\psi, v_0) \in L^2(-r, 0; H) \times H$ be two pairs of initial data for problem (1.1) and u, v as their respective solutions. Defining $w = u - v$, we obtain

$$\begin{aligned}&\frac{d}{dt}w - \nu\Delta w + [(u\cdot\nabla)u - (v\cdot\nabla)v] + \nabla[p_u - p_v] + [\alpha|u|^{\beta-1}u - \alpha|v|^{\beta-1}v]\\&= G[u(t-\rho(t))] - G[v(t-\rho(t))].\end{aligned}$$

We deduce that

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}|w|^2 + \nu\|w\|^2 + [(u\cdot\nabla)u - (v\cdot\nabla)v, w]\\&\qquad + [\alpha|u|^{\beta-1}u - \alpha|v|^{\beta-1}v, w] = \big(G[u(t-\rho(t))] - G[v(t-\rho(t))], w\big).\end{aligned}$$

From [11], there exists a nonnegative constant $\kappa = \kappa(\alpha, \beta)$ such that

$$\alpha\int_\Omega (|u|^{\beta-1}u - |v|^{\beta-1}v)(u-v)dx \geq \kappa\int_\Omega (|u|^{\beta-1} + |v|^{\beta-1})|u-v|^2 dx \geq 0.$$

Therefore, from (2.3), we obtain

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}|w|^2 + \nu\|w\|^2 &\leq c_1|w|\|u\|\|w\| + L_G|w(t-\rho(t))|\cdot|w|\\&\leq c_2|w|^2\|u\|^2 + \nu\|w\|^2 + \frac{1}{2}|w|^2 + \frac{L_G}{2}|w(t-\rho(t))|^2.\end{aligned}$$

Using (1.2), we have

$$\frac{d}{dt}|w|^2 \leq (1 + 2c_2\|u\|^2)|w|^2 + L_G|w(t-\rho(t))|^2.$$

Let $\tau = s - \rho(s)$. Given that $\rho(s) \in [0, r]$ and $\frac{1}{1-\rho'} \leq \frac{1}{1-\rho_*}$, we obtain

$$\begin{aligned}\int_0^t |w(s-\rho(s))|^2 ds &= \int_{0-\rho(0)}^{t-\rho(t)} \frac{1}{1-\rho'}|w(\tau)|^2 d\tau \leq \frac{1}{1-\rho_*}\int_{-r}^t |w(\tau)|^2 d\tau\\&= \frac{1}{1-\rho_*}\int_{-r}^0 |w(\tau)|^2 d\tau + \frac{1}{1-\rho_*}\int_0^t |w(\tau)|^2 d\tau.\end{aligned} \tag{3.1}$$

Using (3.1), we have

$$\begin{aligned}|w(t)|^2 &\leq |w(0)|^2 + \int_0^t (1 + 2c_2\|u(s)\|^2)|w(s)|^2 ds + L_G\int_0^t |w(s-\rho(s))|^2 ds\\&\leq |u_0 - v_0|^2 + \int_0^t (1 + 2c_2\|u(s)\|^2)|w(s)|^2 ds + \frac{L_G}{1-\rho_*}\int_{-r}^0 |w(\tau)|^2 d\tau\\&\quad + \frac{L_G}{1-\rho_*}\int_0^t |w(\tau)|^2 d\tau.\end{aligned}$$

Consequently,

$$\begin{aligned}
&|w(t)|^2\\
&\leq |u_0-v_0|^2+\frac{L_G}{1-\rho_*}\int_{-r}^{0}|\phi-\psi|^2ds+\int_0^t\Big(1+2c_2\|u(s)\|^2+\frac{L_G}{1-\rho_*}\Big)|w(s)|^2ds\\
&\leq |u_0-v_0|^2+\frac{L_G}{1-\rho_*}|\phi-\psi|^2_{L^2(-r,0;H)}+\int_0^t\Big(1+2c_2\|u(s)\|^2+\frac{L_G}{1-\rho_*}\Big)|w(s)|^2ds.
\end{aligned}$$

By the Gronwall lemma, we obtain

$$\begin{aligned}
&|w(t)|^2=|u(t)-v(t)|^2\\
&\leq\Big(|u_0-v_0|^2+\frac{L_G}{1-\rho_*}|\phi-\psi|^2_{L^2(-r,0;H)}\Big)\exp\Big(\int_0^t\Big(1+2c_2\|u(\tau)\|^2+\frac{L_G}{1-\rho_*}\Big)d\tau\Big).
\end{aligned}$$

Here $\theta\in[-r,0]$, we have

$$\begin{aligned}
&|u_t-v_t|^2\\
&\leq\sup_{\theta\in[-r,0]}|u(t+\theta)-v(t+\theta)|^2\\
&\leq\Big(|u_0-v_0|^2+\frac{L_G}{1-\rho_*}|\phi-\psi|^2_{L^2(-r,0;H)}\Big)\exp\Big(\int_0^{t+\theta}\Big(2c_2\|u(s)\|^2+1+\frac{L_G}{1-\rho_*}\Big)ds\Big)\\
&\leq\Big(|u_0-v_0|^2+\frac{L_G}{1-\rho_*}|\phi-\psi|^2_{L^2(-r,0;H)}\Big)\exp\Big(\int_0^{t}\Big(2c_2\|u(s)\|^2+1+\frac{L_G}{1-\rho_*}\Big)ds\Big).
\end{aligned}$$

Thus,

$$\begin{aligned}
|u_t-v_t|^2_{L^2(-r,0;H)}&=\int_{-r}^0|u_t(\theta)-v_t(\theta)|^2d\theta\\
&\leq\int_{-r}^0\sup_{\theta\in[-r,0]}|u(t+\theta)-v(t+\theta)|^2d\theta\\
&\leq\int_{-r}^0\Big(|u_0-v_0|^2+\frac{L_G}{1-\rho_*}|\phi-\psi|^2_{L^2(-r,0;H)}\Big)\\
&\quad\exp\Big(\int_0^{t+\theta}\Big(1+2c_2\|u(s)\|^2+\frac{L_G}{1-\rho_*}\Big)ds\Big)d\theta\\
&\leq\int_{-r}^0\Big(|u_0-v_0|^2+\frac{L_G}{1-\rho_*}|\phi-\psi|^2_{L^2(-r,0;H)}\Big)\\
&\quad\exp\Big(\int_0^{t}\Big(1+2c_2\|u(s)\|^2+\frac{L_G}{1-\rho_*}\Big)ds\Big)d\theta\\
&\leq r\Big(|u_0-v_0|^2+\frac{L_G}{1-\rho_*}|\phi-\psi|^2_{L^2(-r,0;H)}\Big)\\
&\quad\exp\Big(\int_0^{t}\Big(1+2c_2\|u(s)\|^2+\frac{L_G}{1-\rho_*}\Big)ds\Big).
\end{aligned}$$

The proof is completed. □

The following lemma shows the existence of an absorbing set in $L^2(-r,0;H)\times\ H$.

Lemma 3.2. *Let* (1.2) *hold and assume* $u_0, h \in H$ *and* $\phi \in L^2(-r,0;H)$. *Then* $S(t)$ *has an absorbing set* $\mathcal{B}_H$ *in* $L^2(-r,0;H)\times\ H$.

Proof. Multiplying (1.1) to u in H, we have

$$\frac{1}{2}\frac{d}{dt}|u|^2+\nu\|u\|^2+(\alpha|u|^{\beta-1}u,u)=(h,u)+(G(u(t-\rho(t))),u).$$

Using (1.2), we obtain

$$\frac{1}{2}\frac{d}{dt}|u|^2+\nu\|u\|^2+\alpha\int_\Omega|u|^{\beta+1}dx\le|h|\cdot|u|+L_G|u(t-\rho(t))|\cdot|u|.$$

By the Cauchy inequality, we can choose $\sigma>0$ small enough such that $\lambda_1\nu>2L_G+\sigma$. Then,

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}|u|^2+\nu\|u\|^2+\alpha\int_\Omega|u|^{\beta+1}dx\\ \le\ &\frac{1}{4\sigma}|h|^2+\sigma|u|^2+\frac{L_G}{8}|u(t-\rho(t))|^2+2L_G|u|^2.\end{aligned}$$

From $\lambda_1|u|^2\le\|u\|^2$, we obtain

$$\frac{d}{dt}|u|^2\le\frac{1}{2\sigma}|h|^2+\frac{L_G}{4}|u(t-\rho(t))|^2-(2\nu\lambda_1-(2\sigma+4L_G))|u|^2.$$

We choose $m\in(0,m_0)$, $m_0>0$ such that $\nu\lambda_1>\dfrac{L_G}{8(1-\rho_*)}e^{mr}+2L_G+\sigma+\dfrac{m}{2}$. Then,

$$\begin{aligned}\frac{d}{dt}(e^{mt}|u|^2)&=me^{mt}|u|^2+e^{mt}\frac{d}{dt}|u|^2\\&\le me^{mt}|u|^2+\frac{1}{2\sigma}e^{mt}|h|^2+\frac{L_G}{4}e^{mt}|u(t-\rho(t))|^2\\&\quad-(2\nu\lambda_1-(2\sigma+4L_G))e^{mt}|u|^2.\end{aligned}$$

Hence,

$$\frac{d}{dt}(e^{mt}|u|^2)\le\frac{1}{2\sigma}e^{mt}|h|^2+\frac{L_G}{4}e^{mt}|u(t-\rho(t))|^2+(m-(2\nu\lambda_1-(2\sigma+4L_G)))e^{mt}|u|^2.$$

Integrating between 0 and t we get

$$\begin{aligned}e^{mt}|u(t)|^2-|u_0|^2\le\ &\frac{1}{2m\sigma}e^{mt}|h|^2+\frac{L_G}{4}\int_0^t e^{ms}|u(s-\rho(s))|^2ds\\&+(m-(2\nu\lambda_1-(2\sigma+4L_G)))\int_0^t e^{ms}|u(s)|^2ds.\end{aligned}\tag{3.2}$$

Now, let $\tau = s - \rho(s)$. Given that $\rho(s) \in [0, r]$ and $\frac{1}{1-\rho} < \frac{1}{1-\rho_*}$, we then obtain

$$\begin{aligned}\int_0^t e^{ms}|u(s-\rho(s))|^2 ds &\leq \frac{1}{1-\rho_*}\int_{-r}^t e^{m(\tau+r)}u(\tau)d\tau \\ &= \frac{e^{mr}}{1-\rho_*}\int_{-r}^t e^{m\tau}|u(\tau)|d\tau.\end{aligned} \tag{3.3}$$

Combining (3.2) and (3.3), we have

$$\begin{aligned}e^{mt}|u(t)|^2 - |u_0|^2 \leq \frac{1}{2m\sigma}e^{mt}|h|^2 + \frac{L_G}{4(1-\rho_*)}e^{mr}\int_{-r}^t e^{ms}|u(s)|^2 ds \\ + (m - (2\nu\lambda_1 - (2\sigma + 4L_G)))\int_0^t e^{ms}|u(s)|^2 ds.\end{aligned}$$

From $u(t) = \phi(t)$ for $t \in (-r, 0)$, we get

$$\begin{aligned}& e^{mt}|u(t)|^2 - |u_0|^2 \\ \leq\ & \frac{1}{2m\sigma}e^{mt}|h|^2 + \frac{L_G}{4(1-\rho_*)}e^{mr}\int_{-r}^0 e^{ms}|\phi(s)|^2 ds \\ & + \frac{L_G}{4(1-\rho_*)}e^{mr}\int_0^t e^{ms}|u(s)|^2 ds + (m - (2\nu\lambda_1 - (2\sigma + 4L_G)))\int_0^t e^{ms}|u(s)|^2 ds \\ =\ & \frac{1}{2m\sigma}e^{mt}|h|^2 + \frac{L_G}{4(1-\rho_*)}e^{mr}\int_{-r}^0 e^{ms}|\phi(s)|^2 ds \\ & + \Big(m + \frac{L_G}{4(1-\rho_*)}e^{mr} - (2\nu\lambda_1 - (2\sigma + 4L_G))\Big)\int_0^t e^{ms}|u(s)|^2 ds \\ \leq\ & \frac{1}{2m\sigma}e^{mt}|h|^2 + \frac{L_G}{4(1-\rho_*)}e^{mr}\int_{-r}^0 e^{ms}|\phi(s)|^2 ds.\end{aligned}$$

Thus,

$$e^{mt}|u(t)|^2 \leq |u_0|^2 + \frac{1}{2m\sigma}|h|^2 + \frac{L_G}{4(1-\rho_*)}e^{mr}\int_{-r}^0 e^{ms}|\phi(s)|^2 ds.$$

and

$$|u(t)|^2 \leq \frac{1}{2m\sigma}|h|^2 + e^{-mt}\Big(|u_0|^2 + \frac{L_G}{4(1-\rho_*)}e^{mr}\int_{-r}^0 e^{ms}|\phi(s)|^2 ds\Big).$$

Therefore,

$$\begin{aligned}& |u(t-\rho(t))|^2 \\ \leq\ & \frac{1}{2m\sigma}|h|^2 + e^{-m(t-\rho(t))}\Big(|u_0|^2 + \frac{L_G}{4(1-\rho_*)}e^{mr}\int_{-r}^0 e^{ms}|\phi(s)|^2 ds\Big) \\ \leq\ & \frac{1}{2m\sigma}|h|^2 + e^{-mt}\cdot e^{m\rho(t)}\Big(|u_0|^2 + \frac{L_G}{4(1-\rho_*)}e^{mr}\int_{-r}^0 e^{ms}|\phi(s)|^2 ds\Big) \\ \leq\ & \frac{1}{2m\sigma}|h|^2 + e^{-mt}\cdot e^{mr}\Big(|u_0|^2 + \frac{L_G}{4(1-\rho_*)}e^{mr}\int_{-r}^0 e^{ms}|\phi(s)|^2 ds\Big),\end{aligned}$$

for $\rho \in [0, r]$.

Here $\theta \in [-r, 0]$, we obtain

$$
\begin{aligned}
&|u(t+\theta)|^2 \\
&\leq \frac{1}{2m\sigma}|h|^2 + e^{-m(t+\theta)}\Big(|u_0|^2 + \frac{L_G}{4(1-\rho_*)}e^{mr}\int_{-r}^{0} e^{ms}|\phi(s)|^2 ds\Big) \\
&\leq \frac{1}{2m\sigma}|h|^2 + e^{-m(t-r)}\Big(|u_0|^2 + \frac{L_G}{4(1-\rho_*)}e^{mr}\int_{-r}^{0} e^{ms}|\phi(s)|^2 ds\Big) \\
&= \frac{1}{2m\sigma}|h|^2 + e^{-mt}\Big(e^{mr}|u_0|^2 + \frac{L_G}{4(1-\rho_*)}e^{2mr}\int_{-r}^{0} e^{ms}|\phi(s)|^2 ds\Big).
\end{aligned}
$$

Hence,

$$
|u_t|^2 \leq \frac{1}{2m\sigma}|h|^2 + e^{-mt}\Big(e^{mr}|u_0|^2 + \frac{L_G}{4(1-\rho_*)}e^{2mr}\int_{-r}^{0} e^{ms}|\phi(s)|^2 ds\Big).
$$

Denoting $\frac{\rho_H}{2} = \frac{1}{2m\sigma}|h|^2$, we obtain

$$
|u_t|^2 \leq \rho_H. \tag{3.4}
$$

This implies that the ball $B_H = B_{C_H}(0, \rho_H)$ is an absorbing set for the semigroup $S(t)$. $\square$

Lemma 3.3. *Suppose that the assumptions of Lemma 3.2 hold. Then $S(t)$ is asymptotically compact in $L^2(-r, 0; H) \times H$.*

Proof. We denote a bounded set B in $L^2(-r, 0; H)\times H$, and a sequence of solutions $u^n(\cdot)$ in $[0, +\infty)$ with initial data $(u_0^n, \phi^n) \in B$. Consider the sequence $\xi^n = S(t_n)\phi^n$, where $t_n \to +\infty$ as $n \to +\infty$. We will demonstrate that this sequence is relatively compact in $L^2(-r, 0; H) \times H$.

Let $T > 0$, we first prove that ξ^n is relatively compact in $L^2(-r, 0; H)$. From (3.4), there exists an integer n_0 such that $t_n \geq T$ for all $n > n_0$ and

$$
|\xi^n|_{L^2(-r,0;H)} \leq \rho_H. \tag{3.5}
$$

Denote $y^n(\cdot) = u^n_{t_n - T}(\cdot) = u^n(\cdot + t_n - T)$. Then, for each $n \geq 1$ such that $t_n \geq T$, the function y^n is a solution on $[0, T]$ to a problem similar to (1.1), namely,

$$
\frac{d}{dt}y^n(t) - \nu\Delta y^n + (y^n\cdot\nabla)y^n + \alpha|y^n(t)|^{\beta-1}y^n(t) = G(y^n(t-\rho(t))) + h,
$$

where $y_0^n = u^n_{t_n-T}$, $y_T^n = \xi^n$. Then y_0^n satisfies the estimates in (3.5), $\forall n > n_0$. Using similar arguments as in the proof of Theorem 2.1, we obtain

$$
y^n(t_n) \rightharpoonup y(t_0) \text{ weakly in } V \text{ if } t_n \to t_0 \in [0, T].
$$

From (1.2) also, we get

$$\int_0^t |G(y^n - \rho(t))|^2 ds \leq Ct,$$

for all $0 \leq t \leq T$, with $C > 0$ does not depend either on n or t. Using $G(y^n - \rho(t)) \rightharpoonup \xi - \rho(t)$ in $L^2(0, T; H)$, we have

$$\int_s^t |\xi|^2 d\tau \leq \liminf_{n\to+\infty} \int_s^t |G(y_\tau^n - \rho(\tau))| d\tau \leq C(t-s),$$

for all $0 \leq s \leq t \leq T$. Hence, we can take the limit and show that y is a solution to a problem similar to (1.1), that is

$$\frac{d}{dt}(y(t), \zeta) + \nu((y(t), \zeta)) + B(y(t), \zeta) + \int_\Omega \langle (\alpha |y(t)|^{\beta-1} y(t)), \zeta \rangle dx = (\xi, \zeta) + \langle h, \zeta \rangle,$$

for all $\zeta \in L^\infty(0, T; V) \cap L^{\beta+1}(0, T; L^{\beta+1}(\Omega))$. Since

$$\int_s^t \int_\Omega G(z_r - \rho(t)) z(r) dx dr \leq \frac{1}{2\lambda_1 \nu} \int_s^t |G(z_r - \rho(t))|^2 dr + \frac{\lambda_1 \nu}{2} \int_s^t |z(r)|^2 dr,$$

we have the energy inequality

$$\begin{aligned} &|z(t)|^2 + \nu \int_s^t \|z(t)\|^2 dr + 2\int_0^t \langle (\alpha |z(r)|^{\beta-1} z(r)), z(r) \rangle \\ &= |z(s)|^2 + 2\int_s^t \langle h, z(r) \rangle dr + 2C(t-s), \end{aligned}$$

for all $0 \leq s \leq t \leq T$, with $z = y^n$ or $z = y$.

Let J_n and J be two functions from $[0, T]$ to $\mathbb{R}$ and satisfying

$$\begin{aligned} J_n(t) &= \frac{1}{2}|y_n(t)|^2 + \int_0^t \langle (\alpha |y^n(s)|^{\beta-1} y^n(s)), y^n(s) \rangle ds - \int_0^t \langle h, y^n(s) \rangle ds - Ct, \\ J(t) &= \frac{1}{2}|y(t)|^2 + \int_0^t \langle (\alpha |y(s)|^{\beta-1} y(s)), y(s) \rangle ds - \int_0^t \langle h, y(s) \rangle ds - Ct. \end{aligned}$$

Then, J_n, J are two non-increasing continuous functions. Since $y^n(t)$ converges to $y(t)$ for almost every $t \in (0, T)$, we have

$$J_n(t) \to J(t) \text{ for almost every } t \in [0, T].$$

Analogously to the proof of Theorem 2.1, for a fixed $t_0 > 0$, using $\{t_k\}$ with $t_k \to t_0$, we can establish the convergence of the norms:

$$\lim_{n\to\infty} |y^n(t_n)| = |y(t_0)|.$$

Thus, together with the previously established weak convergence, we can conclude that $y^n \to y$ in $C([0,T];H)$.

From $T>0$ and $y^n \to y$ in $C([0,T];H)$, we deduce that $\xi^n \to \varphi$ in $C([0,T];H)$, where $\varphi(s) = y(s+T)$, for $s \in [-r,0]$. By repeating this procedure for $2T, 3T$, etc., and using a diagonal argument (relabelling the subsequence if necessary), we obtain a continuous function $\varphi : (-r,0] \to H$ and a subsequence such that $\xi^n \to \varphi$ in $C([-r,0];H)$ on every interval $[-r,0]$. Moreover, for a fixed $T>0$, we also have

$$|\varphi(s)| \leq \rho_H,$$

for all $s \in [-r,0]$ and $T>0$. We now claim that ξ^n converges to φ in $L^2(-r,0;H)$. Indeed, we need to show that for every $\varepsilon > 0$, there exists n_ε such that

$$|\xi^n(s) - \varphi(s)|^2 \leq \varepsilon, \tag{3.6}$$

for all $n \geq n_\varepsilon$. Fix $T_\varepsilon > 0$ such that $\rho_H^2 \leq \dfrac{\varepsilon}{4}$.

From the above step, we obtain $\xi^n \to \varphi$ in $C([-r,0];H)$, so there exists an $n_\varepsilon = n_\varepsilon(T_\varepsilon)$ such that for all $n \geq n_\varepsilon$, we have

$$|\xi^n(s) - \varphi(s)|^2 \leq \varepsilon, \text{ for al } t_n \geq T_\varepsilon.$$

To prove (3.6), we only need to verify that:

$$|\xi^n(s) - \varphi(s)|^2 \leq \varepsilon, \text{ for all } n \geq n_\varepsilon.$$

Using (3.5) and the choice of T_ε, we can verify that for all $k \in \mathbb{N} \cup \{0\}$ and $s \in [-(T_\varepsilon + k+1), -(T_\varepsilon + k)]$, the following inequality holds

$$\int_{-r}^{0} |\varphi(s)|^2 ds \leq \int_{-r}^{0} |\varphi(s - T_\varepsilon - k)|^2 ds \leq \frac{\varepsilon}{4}.$$

Thus, it suffices to prove that

$$|\xi^n(s)|^2 \leq \frac{\varepsilon}{4}, \forall n \geq n_\varepsilon.$$

We have

$$\xi^n(s) = \begin{cases} \phi^n(s+t_n), & \text{if } s \in [-r, -t_n], \\ u^n(s+t_n), & \text{if } s \in [-t_n, 0]. \end{cases}$$

To complete the proof, we need to establish that

$$\max\Big\{ \int_{-r}^{0} |\phi^n(s+t_n)|^2 ds, \int_{-r}^{0} |u^n(s+t_n)|^2 ds \Big\} \leq \frac{\varepsilon}{4}.$$

The first term of above inequality can be estimated as follows

$$\int_{-r}^{0} |\phi^n(s+t_n)|^2 ds \leq \frac{\varepsilon}{4}.$$

And for the second term, we have

$$\int_{-r}^{0} |u^n(s+t_n)|^2 ds \leq \frac{\varepsilon}{4}.$$

□

We obtain the following theorem from lemmas 3.2 and 3.3.

Theorem 3.1. *Let the assumptions of Lemma 3.2 hold. Then $S(t)$ has a global attractor in $L^2(-r, 0; H) \times H$.*

REFERENCES

[1] Xiaojing Cai & Quansen Jiu, 2008. Weak and strong solutions for the incompressible Navier-Stokes equations with damping. *Journal of Mathematical Analysis and Applications*, 343, 799-809.

[2] Bresch D & Desjardins B, 2003. Existence of global weak solutions for a 2D viscous shallow water equations and convergence to the quasi-geostrophic model. *Communications in Mathematical Physics*, 238(1-2), 211-223.

[3] Bresch D, Desjardins B & Lin CK, 2003. On some compressible fluid models: Korteweg, lubrication, and shallow water systems. *Communications in Partial Differential Equations*, 28(3-4), 843-868.

[4] Hsiao L, 2003. Quasilinear Hyperbolic Systems and Dissipative Mechanisms. *World Scientific*.

[5] Huang FM & Pan RH, 2003. Convergence rate for compressible Euler equations with damping and vacuum. *Archive for Rational Mechanics and Analysis*, 166, 359-376.

[6] Tomas C & Xiaoying H, 2015. A survey on Navier-Stokes models with delays: Existence, uniqueness and asymptotic behavior of solutions. *Discrete and continuous dynamical system series*, 8(6), 1079-1101.

[7] Le TT, 2018. The Existence and uniqueness of solutions to 3D Navier-Stokes equations with damping and delays. *HNUE Journal of Science*, 63(11), 3-9.

[8] Varga K & Sergey Z, 2012. Smooth attractors for the Brinkman-Forchheimer equations with fast growing nonlinearities. *Comunication on Pure and Applied Analysis*, 11(5), 2037-2054.

[9] Chen H, 2012. Asymptotic behavior of stochastic two-dimensional Navier-Stokes equations with delays. *Proceedings of the Indian Academy of Sciences (Mathematical Sciences)*, 122, 283-295.

[10] Constantin P & Foias C, 1988. Navier Stokes equations. *The University of Chicago Press*, Chicago.

[11] Barret JW & Liu WB, 1994. Finite element approximation of the parabolic p-Laplacian. *SIAM Journal on Numerical Analysis*, 31, 413-28.