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ON UNIQUENESS RESULTS OF MEROMORPHIC FUNCTIONS HAVING HYPERORDER LESS THAN ONE

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Abstract. In this paper, we study the relationship between a meromorphic function with hyperorder less than 1 and its exact difference when they share 0 and with counting multiplicities and 1 while ignoring multiplicities, considering truncated multiplicities up to level one. As an application, under the condition of reduced deficiency, we obtain some uniqueness results for such functions. Our results complement existing findings on the uniqueness of meromorphic functions in this research area.

Keywords: meromorphic function, shared values, hyperorder, uniqueness theorem.

1. Introduction

Let f be a non-constant meromorphic function on the complex plane $\mathbb{C}$. The order $\rho(f)$ and hyperorder $\rho_2(f)$ of f are defined as

$$\rho(f) = \limsup_{r\to\infty} \frac{\log^+ T(r,f)}{\log r}, \quad \rho_2(f) = \limsup_{r\to\infty} \frac{\log^+ \log^+ T(r,f)}{\log r},$$

where $T(r,f)$ is the characteristic function of f.

The quality $S(f)$ is denoted by the set of all small functions relative to f. For a positive integer k (possibly $t=+\infty$), a function $a \in S(f)\cup\{\infty\}$, let $E_{t)}(a,f)$ be the set of zeros of $f-a$ with multiplicity at most t, where each zero is counted exactly according to its multiplicity. The corresponding counting function is denoted by

$$N_{t)}\left(r, \frac{1}{f-a}\right).$$

Let $\overline{E}_{t)}(a,f)$ be the set of zeros of $f-a$ with multiplicity at most t, but each zero is counted only once. The corresponding counting function is denoted by

$$\overline{N}_{t)}\left(r, \frac{1}{f-a}\right) \quad \text{or} \quad \overline{N}_{(t}\left(r, \frac{1}{f-a}\right).$$

If $t = +\infty$, we omit the subscript $_{t)}$ from the notation.

Two meromorphic functions f and g are said to share (or share partially) a function $a \in S(f) \cap S(g) \cup \{\infty\}$ IM with truncated multiplicities to level t if

$$\overline{E}_{t)}(a, f) = \overline{E}_{t)}(a, g) \quad (\text{or } \overline{E}_{t)}(a, f) \subseteq \overline{E}_{t)}(a, g)).$$

If

$$E_{t)}(a, f) = E_{t)}(a, g) \quad (\text{or } E_{t)}(a, f) \subseteq E_{t)}(a, g)),$$

then f and g are said to share (or share partially) the function a CM with truncated multiplicities to level t.

There has been significant interest in the uniqueness problem of meromorphic functions $f(z)$ and their first-order exact difference

$$\Delta_c f(z) = f(z + c) - f(z),$$

under certain conditions on their order or hyperorder, particularly when the order is finite or the hyperorder is less than one.

For instance, Chen-Yi [1] established a uniqueness theorem for a transcendental meromorphic function $f(z)$ and its difference $\Delta_c f(z)$ when they share three values CM in the extended complex plane. Their result holds under the assumption that the order $\rho(f)$ is neither an integer nor infinite. Later, Zhang-Liao [2] refined this theorem for transcendental entire functions of finite order.

Subsequently, Lü-Lü [3] extended this result by generalizing the function class from transcendental entire functions to meromorphic functions. In 2018, Chen [4] proved a uniqueness theorem for meromorphic functions of hyperorder less than one, where two values are shared partially CM and one value is shared CM. Later, he and Xu [5] improved this result by considering the case where two values are shared CM and one value is shared IM.

More recently, in 2023, Thuy-Thoan-Nhat [6] gave a simpler proof of this result and strengthened it by excluding all 1-points with multiplicities greater than 2.

The main objective of this paper is to investigate in detail how the uniqueness results mentioned above change when the shared value IM is considered with truncated multiplicities to level one.

Theorem 1.1. *Let f be a nonconstant meromorphic function with hyperorder $\rho_2(f) < 1$. Let $c \in \mathbb{C} \setminus \{0\}$ such that $f(z + c) \not\equiv f(z)$. Assume that $\Delta_c f(z) = f(z + c) - f(z)$ and $f(z)$ share partially $0, \infty$ CM and share 1 IM with truncated multiplicity to level 1, i.e.,*

$$E(0, f) \subseteq E(0, \Delta_c f), \; E(\infty, f) \supseteq E(\infty, \Delta_c f)$$

and

$$\overline{E}_{1)}(1, f) = \overline{E}_{1)}(1, \Delta_c f).$$

Then one of the following cases holds:

(i) $\Delta_c f(z) \equiv f(z)$ *for all* $z \in \mathbb{C}$,

(ii) $\Delta_c f(z) \equiv -2f(z)$ *for all* $z \in \mathbb{C}$.

The following corollaries of Theorem 1.1 give our the uniqueness results in the case of truncated multiplicities to level one.

Corollary 1.1. *If we add the condition that there exists* $a \in S(f) \setminus \{\pm 1, \pm\frac{1}{2}\}$ *such that* $\Theta(a, f) \neq 0$ *then* $\Delta_c f(z) \equiv f(z)$ *for all* $z \in \mathbb{C}$.

Here $\Theta(a, f)$ referred to the reduced deficiency of a with respect to f defined as

$$\Theta(a, f) = 1 - \limsup_{r\to\infty} \frac{\overline{N}\big(r, \frac{1}{f-a}\big)}{T(r, f)}.$$

Corollary 1.2. *With the assumptions of Theorem 1.1, assume in addition that the total reduced deficiency with respect to* f *is not maximal then* $\Delta_c f(z) \equiv f(z)$ *for all* $z \in \mathbb{C}$.

2. Some lemmas

Lemma 2.1. *[7] Let* f *be a nonconstant periodic meromorphic function. Then,* $\rho(f) \geq 1$ *and* $\mu(f) \geq 1$, *where* $\mu(f) = \liminf_{r\to\infty} \frac{\log^+ T(r,f)}{\log r}$ *is the lower order of* f.

Lemma 2.2. *[7] Let* $h(z)$ *be a nonconstant entire function such that* $f(z) = e^{h(z)}$. *Then,* $\rho_2(f) = \rho(h)$.

Lemma 2.3. *[8] Let* $f(z)$ *be a nonconstant meromorphic function of hyperorder* $\rho_2(f) < 1$, *and* $c \in \mathbb{C} \setminus \{0\}$. *Then,*

$$m\big(r, \frac{\Delta_c f}{f}\big) = m\big(r, \frac{f(z+c) - f(z)}{f(z)}\big) = S_1(r, f),$$

where $S_1(r, f) = o(T(r, f)$ *for all* r *outside of a set of finite logarithmic measure.*

Lemma 2.4. *[8] Let* $T : \mathbb{R}^+ \to \mathbb{R}^+$ *be a non-decreasing continuous function and* $s \in (0, +\infty)$ *such that the hyperorder of* T *is strictly less than one, that is*

$$\rho_2(T) = \limsup_{r\to\infty} \frac{\log^+ \log^+ T(r)}{\log r} < 1.$$

Then,

$$T(r + s) = T(r) + o\left(\frac{T(r)}{r^{1-\rho_2-\epsilon}}\right),$$

where $\epsilon > 0$ *and* $r \to \infty$ *outside a subset of finite logarithmic measure.*

3. Proof of Theorems

Proof of Theorem 1.1. Assume that (i) does not hold, i.e., $\Delta_c f \not\equiv f$. We will prove that (ii) must happen, i.e., $\Delta_c f = -2f$. Indeed, by the assumption, we get

$$N\big(r, \frac{1}{f}\big) \le N\big(r, \frac{1}{\Delta_c f}\big).$$

From Lemma 2.3, this inequality implies

$$\begin{aligned} T(r,f) &= T\big(r, \frac{1}{f}\big) + O(1) \\ &= m\big(r, \frac{1}{f}\big) + N\big(r, \frac{1}{f}\big) + O(1) \\ &\le m\big(r, \frac{\Delta_c f}{f}\big) + m\big(r, \frac{1}{\Delta_c f}\big) + N\big(r, \frac{1}{\Delta_c f}\big) + O(1) \\ &= T(r, \Delta_c f) + S_1(r,f). \end{aligned}$$

Then, by Lemma 2.4, we have

$$\begin{aligned} T(r,f) &\le T(r, \Delta_c f) + S_1(r,f) \\ &\le T(r, f_c) + T(r,f) + S_1(r,f) \le 2T(r,f) + S_1(r,f). \end{aligned}$$

So

$$T(r,f) = O(T(r,\Delta_c f)) + S_1(r,f) \text{ and } T(r,\Delta_c f) = O(T(r,f)) + S_1(r,f),$$

which imply that $S(r) := S_1(r,f) = S_1(r, \Delta_c f)$. Put

$$h = \frac{\Delta_c f}{f}.$$

Obviously, $m\big(r,h\big) = S(r)$, h does not have any poles in the whole complex plane. Because of $E(0,f) \subseteq E(0,\Delta_c f)$ and $E(\infty, f) \supseteq E(\infty, \Delta_c f)$, h must be a small function.

By $\overline{E}_{1)}(1,f) = \overline{E}_{1)}(1, \Delta_c f)$, we obtain

$$\overline{N}_{1)}\big(r, \frac{1}{f-1}\big) = \overline{N}_{1)}\big(r, \frac{1}{\Delta_c f - 1}\big) \le N\big(r, \frac{1}{h-1}\big) = S(r). \tag{3.1}$$

We have $f_c = (h+1)f$. By this equation also by Lemma 2.4, we get the same result $S_1(r, f_c) = S_1(r,f) = S(r)$. From inequality (3.1), we deduce that

$$\begin{aligned} \overline{N}_{1)}\big(r, \frac{1}{f - \frac{1}{h}}\big) &= \overline{N}_{1)}\big(r, \frac{1}{\frac{1}{h}(\Delta_c f - 1)}\big) \\ &\le \overline{N}_{1)}\big(r, h\big) + \overline{N}_{1)}\big(r, \frac{1}{\Delta_c f - 1}\big) \le S(r). \end{aligned}$$

Hence,

$$\overline{N}_{1)}\big(r, \frac{1}{f_c-(h+1)}\big) = \overline{N}_{1)}\big(r, \frac{1}{(h+1)(f-1)}\big) \le S(r),$$

and

$$\overline{N}_{1)}\big(r, \frac{1}{f_c-\frac{h+1}{h}}\big) = \overline{N}_{1)}\big(r, \frac{1}{(h+1)(f-\frac{1}{h})}\big) \le S(r).$$

Lemma 2.4 implies that

$$\begin{aligned}\overline{N}_{k)}\big(r, \frac{1}{f_c-a_c}\big) &\le \overline{N}_{k)}\big(r+|c|, \frac{1}{f-a}\big)\\ &\le \overline{N}_{k)}\big(r, \frac{1}{f-a}\big) + o\big(\overline{N}_{k)}\big(r, \frac{1}{f-a}\big)\big) \\ &\le \overline{N}_{k)}\big(r, \frac{1}{f-a}\big) + S_1(r,f)\end{aligned} \tag{3.2}$$

for all $a \in S(f) \cup \{\infty\}$ and $k \in \mathbb{N}^*$. So, applying this inequality to $a = 1$ and by inequality (3.1) again, we get

$$\overline{N}_{1)}\big(r, \frac{1}{f_c-1}\big) \le \overline{N}_{1)}\big(r, \frac{1}{f-1}\big) + S_1(r,f) \le S(r),$$

which implies that

$$\overline{N}_{1)}\big(r, \frac{1}{f-\frac{1}{h+1}}\big) = \overline{N}_{1)}\big(r, \frac{1}{\frac{1}{h+1}(f_c-1)}\big) \le S(r).$$

Notice that $f-a = (f_c-a_c)_{-c}$, hence, using inequality (3.2), we obtain

$$\overline{N}_{k)}\big(r, \frac{1}{f-a}\big) = \overline{N}_{k)}\big(r, \frac{1}{(f_c-a_c)_{-c}}\big) \le \overline{N}_{k)}\big(r, \frac{1}{f_c-a_c}\big) + S_1(r,f).$$

Therefore,

$$\overline{N}_{k)}\big(r, \frac{1}{f-a}\big) = \overline{N}_{k)}\big(r, \frac{1}{f_c-a_c}\big) + S(r), \tag{3.3}$$

for all $a \in S(f) \cup \{\infty\}$ and $k \in \mathbb{N}^*$. We now apply equality (3.3) to $h_{-c}+1$ and $(h_{-c}+1)/h_{-c}$, we get

$$\overline{N}_{1)}\big(r, \frac{1}{f-(h_{-c}+1)}\big) = \overline{N}_{1)}\big(r, \frac{1}{f_c-(h+1)}\big) + S(r) \le S(r),$$

and

$$\overline{N}_{1)}\big(r, \frac{1}{f-\dfrac{h_{-c}+1}{h_{-c}}}\big) = \overline{N}_{1)}\big(r, \frac{1}{f_c-\frac{h+1}{h}}\big) + S(r) \le S(r).$$

Put

$$a_1 = 1, a_2 = \frac{1}{h}, a_3 = \frac{1}{h+1}, a_4 = h_{-c} + 1, a_5 = \frac{h_{-c}+1}{h_{-c}}.$$

For all $i \in \{1, \dots, 5\}$, we have

$$\overline{N}_{1)}\big(r, \frac{1}{f - a_i}\big) \leq S(r).$$

Obviously, a_1 does not be in $\{a_2, a_3, a_4, a_5\}$ and a_2 is difference from a_3.

If $a_2 \equiv a_4$ then $h(h_{-c} + 1) = 1$. It implies that $h(z) \neq 0$ and $h(z) \neq -1$ for all $z \in \mathbb{C}$. Since h is entire, by Picard theorem, h is constant and hence $h = \frac{-1\pm\sqrt{5}}{2}$. On the other hand, $h + 1 = \frac{1}{h} = \frac{f_c}{f}$ implies that $f = hf_c$. So applying (3.3) again, we have

$$\overline{N}_{1)}\big(r, \frac{1}{f-h}\big) = \overline{N}_{1)}\big(r, \frac{1}{h(f_c - 1)}\big) \leq \overline{N}_{1)}\big(r, \frac{1}{f-1}\big) + S(r) \leq S(r),$$

and

$$\overline{N}_{1)}\big(r, \frac{1}{f-h^2}\big) = \overline{N}_{1)}\big(r, \frac{1}{h(f_c - h)}\big) \leq \overline{N}_{1)}\big(r, \frac{1}{f-h}\big) + S(r) \leq S(r).$$

Similarly, we have

$$\overline{N}_{1)}\big(r, \frac{1}{f-h^n}\big) \leq S(r),$$

for all $n \in \mathbb{N}$. Set $b_1 = 1, b_2 = h, b_3 = h^2, b_4 = h^3, b_5 = h^4$. It is easy to see that $b_i \neq b_j$ with $i \neq j$. Applying Second main theorem, we get

$$\begin{aligned}
3T(r,f) &\leq \sum_{i=1}^{5} \overline{N}\big(r, \frac{1}{f-b_i}\big) + S(r) \\
&= \sum_{i=1}^{5} \left(\overline{N}_{1)}\big(r, \frac{1}{f-b_i}\big) + \overline{N}_{(2}\big(r, \frac{1}{f-b_i}\big) \right) + S(r) \\
&\leq \frac{1}{2} \sum_{i=1}^{5} N\big(r, \frac{1}{f-b_i}\big) + S(r) \\
&\leq \frac{5}{2} T\big(r,f\big) + S(r).
\end{aligned}$$

This is a contradiction. Here $\overline{N}_{(2}\big(r, \frac{1}{f-b_i}\big)$ is denoted by the counting function corresponding to $\overline{E}_{(2}(b_i, f) = \overline{E}(b_i, f) \setminus \overline{E}_{1)}(b_i, f)$. Therefore, $a_2 \neq a_4$.

If $a_2 \equiv a_5$, then $h_{-c} = h(h_{-c} + 1)$, which implies that $h = h_c(h+1) = h_c \frac{f_c}{f}$. So we have $h_c f_c = hf$. Clearly, $h + 1$ has not any zeros and hence $h + 1$ is in the form $h + 1 = e^{\tau}$. Put $F = hf$ then $F_c = F$, we obtain

$$\overline{N}_{1)}\big(r, \frac{1}{F-1}\big) = \overline{N}_{1)}\big(r, \frac{1}{h(f - \frac{1}{h})}\big) \leq S(r),$$

and

$$\overline{N}_{1)}\big(r, \frac{1}{F-h}\big) = \overline{N}_{1)}\big(r, \frac{1}{h(f-1)}\big) \leq S(r).$$

From this inequality and by applying (3.3) again, we get

$$\overline{N}_{1)}\big(r, \frac{1}{F-h_c}\big) = \overline{N}_{1)}\big(r, \frac{1}{F_c-h_c}\big) = \overline{N}_{1)}\big(r, \frac{1}{F-h}\big) + S(r) \leq S(r)$$

and hence, we have

$$\overline{N}_{1)}\big(r, \frac{1}{F-h_{nc}}\big) \leq S(r)$$

for all $n \in \mathbb{N}$.

If $h = h_{nc}$ for some n, then $h+1 = h_{nc}+1$, which implies that $e^{\tau} = e^{\tau_{nc}}$. Then, we have $\tau' = \tau'_{nc}$. If τ' is not constant, then by Lemmas 2.1 and 2.2, we have

$$1 > \rho_2(h) = \rho(\tau) \geq \rho(\tau') \geq 1.$$

This is a contradiction, so τ' is constant and if $\tau' \neq 0$, then $h+1 = e^{\frac{2m\pi i}{nc}z+b}$, with some $m \in \mathbb{Z}$ and $b \in \mathbb{C}$. Then since $h = h_c(h+1)$, we have

$$h = h_c(h+1) = (e^{\frac{2m\pi i}{nc}(z+c)+b} - 1)(h+1) = (e^{\frac{2m\pi i}{n}}(h+1) - 1)(h+1),$$

which implies h must be constant. This is a contradiction. Thus, $\tau' = 0$, i.e., also h is constant. This is a contradiction to the condition $h = h_c(h+1)$. Therefore, $h \neq h_{nc}$ for all $n \in \mathbb{N}$. Applying the Second main theorem to F with $1, h, h_c, h_{2c}, h_{3c}$, we get a contradiction. Therefore, we have $a_2 \neq a_5$.

Case 1: $a_3 \neq a_4$. If $a_3 \equiv a_5$ then $h_{-c} = (h_{-c}+1)(h+1)$ or $h = (h+1)(h_c+1)$. These imply that $h(z) \neq -1$ and $h(z) \neq 0$. By Picard's theorem again, h must be a constant function and hence $h^2 + h + 1 = 0$. This implies that $h^3 = 1$, $-h^2 = h+1 = \frac{f_c}{f}$ and $f_c = -h^2 f$. By using the same argument as in the case of $a_2 = a_4$, we get

$$\overline{N}\big(r, \frac{1}{f-c_i}\big) \leq S(r) \ (1 \leq i \leq 5),$$

where $c_1 = 1, c_2 = -h^2, c_3 = h^4 = h, c_4 = -h^6 = -1, c_5 = h^8 = h^2$. Clearly, c_i $(1 \leq i \leq 5)$ are distinct. Then, by applying the Second main theorem to f with c_i $(1 \leq i \leq 5)$, we get a contradiction. Thus $a_3 \neq a_5$. Obviously $a_4 \neq a_5$ and so a_i $(1 \leq i \leq 5)$ are distinct. Applying the Second main theorem again to f with those functions, we get a contradiction again.

Case 2: $a_3 \equiv a_4$. Then $(h_{-c}+1)(h+1) = 1$, which implies $h+1$ has not any zeros and hence $h+1 = e^{\tau}$. So we have $e^{\tau+\tau_{-c}} = 1$. It follows that $\tau' + \tau'_{-c} = 0$ and then $\tau' = \tau'_{2c}$. By using the same argument above, we obtain τ' is constant and hence τ and h is constant also. Since the condition of h, we deduce that $h+1 = -1$, hence $\Delta_c f = -2f$, i.e., (ii) happens. The proof the theorem is completed. □

Proof of Corollaries 1.1 and 1.2. Assume that $\Delta_c f \neq f$. Then, by Theorem 1.1 and its proof, we have $\Delta_c f = -2f$ and $a_1 = 1, a_2 = -\frac{1}{2}, a_3 = a_4 = -1, a_5 = \frac{-1}{2}$. Second main theorem implies that

$$\begin{aligned} 2T(r,f) &\leq \sum_{i=1, i\neq 4}^{5} \overline{N}\big(r, \frac{1}{f-a_i}\big) + S(r) \\ &= \sum_{i=1, i\neq 4}^{5} \Big(\overline{N}_{1)}\big(r, \frac{1}{f-a_i}\big) + \overline{N}_{(2}\big(r, \frac{1}{f-a_i}\big)\Big) + S(r) \\ &\leq \frac{1}{2}\sum_{i=1, i\neq 4}^{5} N\big(r, \frac{1}{f-a_i}\big) + S(r) \leq 2T\big(r,f\big) + S(r), \end{aligned}$$

which implies $\sum_{i=1, i\neq 4}^{5} \Theta(a_i, f) = 2$, i.e., the total of the reduced deficiency with respect to f is maximal, hence, $\Theta(a,f) = 0$ for all $a \notin \{a_1, a_2, a_3, a_5\}$. These contradict the assumptions of Corollaries 1.1 and 1.2. Hence $\Delta_c f(z) = f(z)$ and the corollaries are proved. □

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