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AN INTEGRAL REPRESENTATION OF THE PELL NUMBERS AND THE PELL-LUCAS NUMBERS

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Abstract. We report on an integral representation for the Pell sequence, Pell-Lucas sequence, Balancing sequence and Lucas-Balancing sequence. This integral representation is based on the generating function and the Binet-like formulas of the aforementioned sequences. Many other integral representations of these numbers can be found by applying the other known relations between them.
Keywords: Pell sequence, Pell-Lucas sequences, integral representations.

1. Introduction

In 2015, Glasser and Zhou [1] proposed an integral representation for the Fibonacci sequence based on a Fourier integral representation, for producing this uniform asymptotic expansion.

$$F_n = \frac{1}{\sqrt{5}}\left(\frac{\sqrt{5}+1}{2}\right)^n - \frac{2}{\pi}\int_0^{\infty} \frac{\sin(x/2)}{x}\frac{\cos(nx) - 2\sin(nx)\sin x}{5\sin^2 x + \cos^2 x}dx$$

In 2023, Stewart [2] derived a simpler integral representation for this sequence.

$$F_n = \frac{n}{2^n}\int_{-1}^{1} \left(1 + x\sqrt{5}\right)^{n-1} dx;$$

In this article, we formulate a simple integral representation for some other famous sequences such as Pell sequence, Pell-Lucas sequence, Balancing sequence and Lucas-Balancing sequence and give some links between these numbers.

2. Auxiliary results

The Pell sequence $\{P_n\}_{n=0}^{\infty}$ and the Pell - Lucas sequence $\{Q_n\}_{n=0}^{\infty}$ are defined by the recursions

$$P_0 = 0, P_1 = 1; P_n = 2P_{n-1} + P_{n-2}, n \geq 2;$$

and

$$Q_0 = 1, Q_1 = 1; Q_n = 2Q_{n-1} + Q_{n-2}, n \geq 2.$$

The Balancing sequence $\{B_n\}_{n=0}^{\infty}$ and the Lucas - Balancing sequence $\{C_n\}_{n=0}^{\infty}$ are defined by the recursions

$$B_0 = 0, B_1 = 1; B_n = 6B_{n-1} - B_{n-2}, n \geq 2;$$

and

$$C_0 = 1, C_1 = 3; C_n = 6C_{n-1} - C_{n-2}, n \geq 2.$$

Then, we have some well-known facts on the sequences $(P_n)_{n\geq 0}$, $(Q_n)_{n\geq 0}$, and also $(B_n)_{n\geq 0}$, $(C_n)_{n\geq 0}$, which we have the following *Binet-like formulas* for these numbers as the following lemma.

Lemma 2.1 ([3]). *Let n be a nonnegative integer. Then, we have*

a) $P_n = \dfrac{\alpha_1^n - \alpha_2^n}{2\sqrt{2}}$;

b) $Q_n = \dfrac{\alpha_1^n + \alpha_2^n}{2}$;

c) $B_n = \dfrac{\beta_1^n - \beta_2^n}{4\sqrt{2}}$;

d) $C_n = \dfrac{\beta_1^n + \beta_2^n}{2}$.

where $\alpha_1 = 1 + \sqrt{2}$ and $\alpha_2 = 1 - \sqrt{2}$ are two roots of characteristic equation of the Pell recurrence $x^2 - 2x - 1 = 0$; and $\beta_1 = 3 + 2\sqrt{2}$ and $\beta_2 = 3 - 2\sqrt{2}$ are two roots of characteristic equation of the balancing recurrence $x^2 - 6x + 1 = 0$.

From the Binet-like formulas of P_n, Q_n, B_n and C_n, we can easily obtain the following important results.

$$Q_n = P_n + P_{n-1}; \text{and } C_n = 3B_n - B_{n-1}.$$

Now, we consider the generating function of the Pell numbers.

$$P(x) = \sum_{n=0}^{\infty} P_n x^n.$$

Then, we have

Proposition 2.1.
$$P(x) = \frac{x}{1-2x-x^2}.$$

Proof. We have

$$\begin{aligned}
(1-2x-x^2)\sum_{n=0}^{\infty} P_n x^n &= \sum_{n=0}^{\infty} P_n x^n - 2\sum_{n=0}^{\infty} P_n x^{n+1} - \sum_{n=0}^{\infty} P_n x^{n+2} \\
&= \sum_{n=0}^{\infty} P_n x^n - 2\sum_{n=1}^{\infty} P_{n-1} x^n - \sum_{n=2}^{\infty} P_{n-2} x^n \\
&= P_0 + P_1 x - 2P_0 x + \sum_{n=2}^{\infty} (P_n - 2P_{n-1} - P_{n-2}) x^n \\
&= x
\end{aligned}$$

Thus, we obtain $P(x) = \dfrac{x}{1-2x-x^2}$. □

Similarly, we obtain the formulas for generating function of the Pell - Lucas numbers, the Balancing numbers, and the Lucas-Balancing numbers, which are given by the following proposition.

Proposition 2.2. *Let $Q(x), B(x)$ and $C(x)$ be the generating function of the Pell-Lucas numbers, the balancing numbers, and the Lucas-Balancing numbers respectively. Then, we have*

a) $Q(x) = \sum\limits_{n=0}^{\infty} Q_n x^n = \dfrac{1-x}{1-2x-x^2}.$

b) $B(x) = \sum\limits_{n=0}^{\infty} B_n x^n = \dfrac{x}{1-6x+x^2}.$

c) $C(x) = \sum\limits_{n=0}^{\infty} C_n x^n = \dfrac{1-3x}{1-6x+x^2}.$

3. An integral representation of the Pell and Pell - Lucas numbers

In this section, we give an integral representation of the Pell and Balancing numbers as the following theorem.

Theorem 3.1. *Let n be a nonnegative integer. Then, we have*

a) $P_n = \dfrac{n}{2}\int\limits_{-1}^{1} \left(1+x\sqrt{2}\right)^{n-1} dx.$

b) $Q_n = \dfrac{1}{2}\int\limits_{-1}^{1} \left[n\left(2+x\sqrt{2}\right)-1\right]\left(1+x\sqrt{2}\right)^{n-2} dx.$

c) $B_n = \frac{n}{2} \int\limits_{-1}^{1} \left(3 + 2x\sqrt{2}\right)^{n-1} dx.$

d) $C_n = \frac{1}{2} \int\limits_{-1}^{1} \left[n\left(8 + 6x\sqrt{2}\right) + 1\right] \left(3 + 2x\sqrt{2}\right)^{n-2} dx.$

Proof. By Lemma 2.1. a), we have

$$\begin{aligned} P_n &= \frac{\alpha_1^n - \alpha_2^n}{2\sqrt{2}} \\ &= \frac{1}{2\sqrt{2}} \left[\left(1+\sqrt{2}\right)^n - \left(1-\sqrt{2}\right)^n\right] \\ &= \frac{1}{2\sqrt{2}} \left(1 + x\sqrt{2}\right)^n \Big|_{-1}^{1} \\ &= \frac{1}{2\sqrt{2}} \int_{-1}^{1} n\left(1+x\sqrt{2}\right)^{n-1} d\left(1+x\sqrt{2}\right) \\ &= \frac{n}{2} \int_{-1}^{1} \left(1+x\sqrt{2}\right)^{n-1} dx. \end{aligned}$$

□

Similarly, we also obtain the integral representation for Q_n, B_n and C_n.

As an application of the integral representation given by Theorem (3.1), we will derive integral representations of the generating function of the Pell numbers.

Theorem 3.2. *We have*

$$P(x) = \frac{x}{2} \sum_{n=0}^{\infty} \int_{-1}^{1} n\left[\left(1+t\sqrt{2}\right)x\right]^{n-1} dt.$$

Proof. Replacing P_n in the formal series with its integral representation, we have

$$\begin{aligned} P(x) = \sum_{n=0}^{\infty} P_n x^n &= \sum_{n=0}^{\infty} \frac{n}{2} \int_{-1}^{1} \left(1+t\sqrt{2}\right)^{n-1} x^n dt \\ &= \frac{x}{2} \sum_{n=0}^{\infty} \int_{-1}^{1} n\left[\left(1+t\sqrt{2}\right)x\right]^{n-1} dt. \end{aligned}$$

□

From Theorem 3.1, we have

$$\begin{aligned} P_{2n} &= \frac{2n}{2}\int_{-1}^{1}\left(1+x\sqrt{2}\right)^{2n-1}dx \\ &= n\int_{-1}^{1}\left(1+x\sqrt{2}\right)^{2n-1}dx. \end{aligned}$$

Substituting $1+x\sqrt{2}=\sqrt{3+2t\sqrt{2}}$, we obtain $dx=\dfrac{dt}{\sqrt{3+2t\sqrt{2}}}$. Therefore

$$\begin{aligned} P_{2n} &= n\int_{-1}^{1}\left(1+x\sqrt{2}\right)^{2n-1}dx \\ &= n\int_{-1}^{1}\left(\sqrt{3+2t\sqrt{2}}\right)^{2n-1}\frac{dt}{\sqrt{3+2t\sqrt{2}}} \\ &= n\int_{-1}^{1}\left(3+2t\sqrt{2}\right)^{n-1}dt. \end{aligned}$$

So, we can easily see that $P_{2n}=2B_n$, thus

$$\sum_{n=0}^{\infty}P_{2n}x^n = 2\sum_{n=0}^{\infty}B_nx^n = \frac{2x}{1-6x+x^2}.$$

Now, we will end this by giving one more formula of Pell numbers, it is almost like the Catalan formula for Pell numbers.

Theorem 3.3. *Let n be positive integer. Then, we have*

$$P_n = \sum_{k=0}^{\left\lfloor \frac{n-1}{2}\right\rfloor}\binom{n}{2k+1}2^k,$$

where $\lfloor x\rfloor$ is the greatest integer less than or equal to x.

Proof. We have,

$$
\begin{aligned}
P_n &= \frac{n}{2}\int\limits_{-1}^{1}\left(1+x\sqrt{2}\right)^{n-1}dx \\
&= \frac{n}{2}\int\limits_{-1}^{1}\sum_{k=0}^{n-1}\binom{n-1}{k}\left(x\sqrt{2}\right)^{k}dx \\
&= \frac{n}{2}\binom{n-1}{k}2^{\frac{k}{2}}\sum_{k=0}^{n-1}\int\limits_{-1}^{1}x^k dx.
\end{aligned}
$$

Note that

$$
\int\limits_{-1}^{1}x^k dx = \begin{cases} 0, & \text{k odd} \\ \dfrac{2}{k+1}, & \text{k even,} \end{cases}
$$

and

$$
\binom{n-1}{k}\frac{n}{k+1} = \binom{n}{k+1}.
$$

Thus,

$$
\begin{aligned}
P_n &= \frac{n}{2}\binom{n-1}{k}2^{\frac{k}{2}}\sum_{k=1}^{n-1}\int\limits_{-1}^{1}x^k dx \\
&= \sum_{\substack{k=0\\2|k}}^{2\left\lfloor\frac{n-1}{2}\right\rfloor}\frac{n}{2}\binom{n-1}{k}2^{\frac{k}{2}}\frac{2}{k+1} \\
&= \sum_{\substack{k=0\\2|k}}^{2\left\lfloor\frac{n-1}{2}\right\rfloor}\frac{n}{k+1}\binom{n-1}{k}2^{\frac{k}{2}} \\
&= \sum_{\substack{k=0\\2|k}}^{2\left\lfloor\frac{n-1}{2}\right\rfloor}\binom{n}{k+1}2^{\frac{k}{2}}.
\end{aligned}
$$

So, we deduce that

$$
P_n = \sum_{k=0}^{\left\lfloor\frac{n-1}{2}\right\rfloor}\binom{n}{2k+1}2^k.
$$

$\square$

Many other integral representations of these numbers can be found by applying known relations between P_n, Q_n, B_n, C_n with the assumption of $\alpha_1^2 = \beta_1$ and $\alpha_2^2 = \beta_2$, which the reader may care to find. Here are some links to these numbers.

Theorem 3.4. *Let n be positive integer. Then, we have*

a) $2B_n = P_{2n}; C_n = Q_{2n}$.

b) $B_n = P_n Q_n$.

c) $B_n = \dfrac{P_{2n-1} + Q_{2n-1}}{2}$.

d) $B_n = \sum\limits_{k=1}^{n} P_{2k-1}$.

e) $\dfrac{C_n - 1}{2} = \sum\limits_{k=1}^{n} Q_{2k-1}$.

f) $2B_n - 1 = \sum\limits_{k=1}^{2n-1} Q_k$.

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