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THE NORMALITY OF THE FAMILY OF HOLOMORPHIC MAPPINGS INTO THE COMPLEMENT OF A NEIGHBORHOOD OF A HYPERSURFACE IN THE COMPLEX PROJECTIVE SPACE

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Abstract. In 1907, Paul Montel raised the concept of normal family of meromorphic functions. The high-dimensional version of Montel theorem was first studied by Tu Zhenhan in 1999. In this paper, we examine the normality of the family of holomorphic mappings from a domain D in $\mathbb{C}^m$ into the complement of the $\epsilon-$neighborhood of a hypersurface in the complex projective space $P^n(\mathbb{C})$ (with the Fubini-Study distance d_{FS}).
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1. Introduction

A family $\mathcal{F}$ of holomorphic mappings of a domain $D \subset \mathbb{C}^m$ into $P^n(\mathbb{C})$ is said to be normal on D if any sequence in $\mathcal{F}$ has a holomorphically convergent subsequence on D.

Perhaps the most celebrated theorem in the theory of normal families is the following criterion of Montel [1].

Theorem 1.1. *Let $\mathcal{F}$ be a family of meromorphic functions in a domain $D \subset \mathbb{C}$, and let a, b, c be three distinct points in $\widehat{\mathbb{C}}$. Assume that all functions in $\mathcal{F}$ omit three points a, b, c in D. Then $\mathcal{F}$ is a normal family in D.*

For the high dimensional case, some types of Montel's theorem have been established. We refer the readers to [2]-[4].

The purpose of this paper is to examine the case of having one hypersurface (rather than having $2n + 1$ hypersurfaces).

Our main result can be stated as follows.

Theorem 1.2. *Let n, m, d be positive integers. Let $\mathcal{F}$ be a family of holomorphic mappings of a domain $D \subset \mathbb{C}^m$ into $P^n(\mathbb{C})$. For each $f \in \mathcal{F}$ we consider a hypersurface H_f in $P^n(\mathbb{C})$ of degree d. Assume that for each compact subset K in D, there exists a positive constant $\epsilon(K)$ such that $d_{FS}(f(z), H_f) \geq \epsilon(K)$ for all $z \in K$ and $f \in \mathcal{F}$. Then, $\mathcal{F}$ is normal on D.*

2. Notations

In $P^n(\mathbb{C})$ with the homogeneous coordinates $\omega = (\omega_0 : \cdots : \omega_n)$, the Fubini-Study metric is given by

$$ds^2 = \frac{\langle d\omega, d\omega \rangle \langle \omega, \omega \rangle - |\langle \omega, d\omega \rangle|^2}{\langle \omega, \omega \rangle^2}$$

where $\langle \cdot, \cdot \rangle$ stands for the standard Hermitian product in $\mathbb{C}^{n+1}$. Denote by d_{FS} the Fubini-Study distance in $P^n(\mathbb{C})$. If $\overrightarrow{u}$ and $\overrightarrow{v}$ are two unit vectors in $\mathbb{C}^{n+1}$ representing points U, V in $P^n(\mathbb{C})$ then, the Fubini-Study distance between U and V is

$$d_{FS}(U, V) = \|\overrightarrow{u} \wedge \overrightarrow{v}\|.$$

The distance from each point $p \in \mathbb{P}^n(\mathbb{C})$ to a set $S \subset P^n(\mathbb{C})$ is defined by $d_{FS}(p, S) := \inf\{d_{FS}(p, q) : q \in S\}$ For each positive constant ϵ and each non-empty subset $S \subset P^n(\mathbb{C})$, the ϵ-neighborhood of S, denoted by S_ϵ, is the set of points whose Fubini-Study distance to S is less than ϵ.

3. Proof of Theorem 1.2

Lemma 3.1 ([5], Corollary 3.1)**.** *Let $\mathcal{F}$ be a family of holomorphic mappings of a domain $D \subset \mathbb{C}^m$ into $P^n(\mathbb{C})$. If $\mathcal{F}$ is not normal then there exist sequences $\{z_k\} \subset D$ with $z_k \to z_0 \in D$, $\{f_k\} \subset \mathcal{F}$, $\{\rho_k\} \subset \mathbb{R}$ with $\rho_k \to 0^+$, $\{u_k\} \subset \mathbb{C}^m$ of Euclidean unit vectors, such that $g_k(\zeta) := f_k(z_k + \rho_k u_k \zeta)$ converges uniformly on compact subsets of $\mathbb{C}$ to a nonconstant holomorphic mapping g of $\mathbb{C}$ into $P^n(\mathbb{C})$.*

Lemma 3.2 ([6], Corollary 3.7)**.** *Let D be a hypersurface in $P^n(\mathbb{C})$, and let ϵ be a positive constant. Then every entire curve f in $P^n(\mathbb{C}) \setminus H_\epsilon$ is constant.*

Proof of Theorem 1.2. Suppose that $\mathcal{F}$ is not normal, by Lemma 3.1, there exist sequences $\{z_k\} \subset D$ with $z_k \to z_0 \in D$, $\{f_k\} \subset \mathcal{F}$, $\{\rho_k\} \subset \mathbb{R}$ with $\rho_k \to 0^+$, and Euclidean unit vectors $u_k \in \mathbb{C}^m$ such that $g_k(\zeta) := f_k(z_k + \rho_k u_k \zeta)$, where $\zeta \in \mathbb{C}$ satisfies $z_k + \rho_k u_k \zeta \in D$, converges uniformly on compact subsets of $\mathbb{C}$ to a nonconstant holomorphic mapping g of $\mathbb{C}$ into $P^n(\mathbb{C})$.

We take a reduced representation $\widehat{f_k} = (f_{0k}, \ldots, f_{nk})$ of f_k. Take a closed ball $B(z_0, R) := \{z : \|z - z_0\| \leq R\} \subset D$. By the assumption, there is a positive constant ϵ

such that

$$d_{FS}(f_k(z), H_{f_k}) \geq \epsilon \tag{3.1}$$

for all $z \in B(z_0, R)$ and all $k \in \mathbb{N}$.

By the compactness of the complex projective space, by replacing by a subsequence if necessary, we may assume $\{H_{f_k}\}$ converges to a hypersurface H of degree d.

For any fixed point $\zeta \in \mathbb{C}$, we have $z_k + \rho_k u_k \zeta \in B(z_0, R)$ for all large integer k. Therefore, we have

$$\begin{aligned} d_{FS}(g(\zeta), H) &= \lim_{k\to\infty} d_{FS}(g_k(\zeta), H_{f_k}) \\ &= \lim_{k\to\infty} (f_k(z_k + \rho_k u_k \zeta), H_{f_k}) \geq \epsilon. \end{aligned}$$

Hence, by Lemma 3.2, g is a constant curve. This is impossible. The proof of Theorem 1.2 is completed. □

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