HNUE JOURNAL OF SCIENCE
Natural Science, 2025, Volume 70, Issue 1, pp. 3-14
This paper is available online at http://hnuejs.edu.vn/ns
DOI: 10.18173/2354-1059.2025-0001

NONEXISTENCE RESULT FOR QUASILINEAR ELLIPTIC INEQUALITIES INVOLVING THE GRUSHIN OPERATOR

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Received February 14, 2025. Revised March 20, 2025. Accepted March 31, 2025.

Abstract. In this paper, we establish a nonexistence result of positive solutions of the inequality $-\text{div}_G(|\nabla_G u|^{p-2}\nabla_G u) \geq u^q \quad \text{in } \mathbb{R}^N = \mathbb{R}^{N_1} \times \mathbb{R}^{N_2}$, where $\nabla_G = (\nabla_x, |x|^a \nabla_y)$ is the Grushin gradient, $p, q > 1$. Here $(x, y) \in \mathbb{R}^{N_1} \times \mathbb{R}^{N_2}$ and $a > 0$. As a generalization of this result, we also establish a nonexistence result of positive solutions to the inequality $-\text{div}_G\Big(A(|\nabla_G u|)\nabla_G u\Big) \geq u^q \quad \text{in } \mathbb{R}^N = \mathbb{R}^{N_1} \times \mathbb{R}^{N_2}$, where $A : \mathbb{R}_+ \to \mathbb{R}_+$ satisfies some conditions below. Our results can be seen as a generalization of that in [Mitidieri E & Pohozaev SI, (2001). A priori estimates and the absence of solutions of nonlinear partial differential equations and inequalities. *Tr. Mat. Inst. Steklova*, 234, 1-384] from the Laplace operator to the Grushin operator.
Keywords: Liouville-type theorem, quasilinear inequality, Grushin operator, nonexistence result.

1. Introduction and main results

The nonexistence result is an important issue in the theory of partial differential equations. In this paper, we first study some nonexistence results for quasilinear elliptic inequality involving Grushin-type operator

$$-\text{div}_G(|\nabla_G u|^{p-2}\nabla_G u) \geq u^q \quad \text{in } \mathbb{R}^N = \mathbb{R}^{N_1} \times \mathbb{R}^{N_2}. \tag{1.1}$$

We split $\mathbb{R}^N = \mathbb{R}^{N_1} \times \mathbb{R}^{N_2}$, and for each $z \in \mathbb{R}^N$, we write $z = (x, y)$, where $x \in \mathbb{R}^{N_1}, y \in \mathbb{R}^{N_2}$. The Grushin gradient is defined as

$$\nabla_G = (\nabla_x, |x|^a \nabla_y),$$

where a is a non-negative scalar and ∇_x, ∇_y are the standard Euclidean gradients in $\mathbb{R}^{N_1}, \mathbb{R}^{N_2}$, respectively. Then, we have $\text{div}_G = \nabla_G \cdot$, and

$$\Delta_G = \text{div}_G \circ \nabla_G = \Delta_x + |x|^{2a}\Delta_y$$

is called the Grushin operator. The homogeneous dimension in $\mathbb{R}^N$ associated to the Grushin operator is

$$Q = N_1 + (a+1)N_2. \tag{1.2}$$

First, we consider inequality (1.1) in the special case $p = 2$. If $a = 0$, then (1.1) is reduced to

$$-\Delta u \geq u^q \text{ in } \mathbb{R}^N.$$

The existence and nonexistence of positive solutions of this inequality was established in [1] (see also [2] for $q > 1$), where the critical exponent is given by $q_c = \frac{N}{N-2}$. In the case of Grushin operator, DAmbrosio and Mitidieri [3] established the nonexistence of positive solutions under the condition $0 < q \leq \frac{Q}{Q-2}$ (see also [4] or [5] for $1 < q \leq \frac{Q}{Q-2}$), where Q is the homogeneous dimension of $\mathbb{R}^N$ associated to the Grushin operator.

Now, we consider the general case of $p > 1$. If $a = 0$, then the optimal nonexistence result for positive solutions of the inequality

$$-\operatorname{div}\left(|\nabla u|^{p-2}\nabla u\right) \geq u^q \text{ in } \mathbb{R}^N \tag{1.3}$$

was shown in [6]. To this case, the nonexistence of positive solutions of (1.3) was established when

(i) $0 < p-1 < q \leq \frac{N(p-1)}{N-p}$ and $N > p$, or

(ii) $0 \leq q \leq p-1, p > 1$ and $N > 1$.

Our purpose in this paper is to extend this result to the Grushin framework which, to the best of our knowledge, has not been considered in the literature. Notice that the Laplace operator can be regared as a special case of the Grushin operator with $a = 0$. When $a > 0$, Δ_G is only elliptic when $|x| \neq 0$. The Grushin operator Δ_G was introduced in [7], [8] and has received a lot of attention of researchers. It is well-known that the operator Δ_G belongs to a wider class of subelliptic operators which has been studied by many authors [9]-[19].

To state our results, we introduce a functional class of solutions for (1.1). Let $\sigma < 0$ be a small enough scalar, $p > 1$ and $q \geq 0$, we define

$$W^{1,p}_{\sigma,G,\mathrm{loc}}(\mathbb{R}^N) = \{u : \mathbb{R}^N \to \mathbb{R}_+, u^{q+\sigma}, |\nabla_G u|^p u^{\sigma-1} \in L^1_{\mathrm{loc}}(\mathbb{R}^N)\}.$$

Our first result is presented as in the following theorem.

Theorem 1.1. *If either*

(i) $0 < p-1 < q \leq \frac{Q(p-1)}{Q-p}$ *and* $Q > p$, *or*

(ii) $0 \leq q \leq p-1, p > 1$ *and* $Q > 1$, *where* Q *is defined as in* (1.2)*, then the problem* (1.1) *has no nontrivial non-negative solutions of class* $W^{1,p}_{\sigma,G,\mathrm{loc}}(\mathbb{R}^N)$*.*

Note that, when $a = 0$, we have $Q = N$ and we reobtain the result in [2] from Theorem 1.1.

The following result is an extension of Theorem 1.1.

Theorem 1.2. *Let* $p > 1$ *and* $A : \mathbb{R}_+ \to \mathbb{R}_+$ *be a function satisfying: there exist* $c_1, c_2 > 0$ *such that*

$$c_1 t^{p-1} \le A(t)t \le c_2 t^{p-1}, \tag{1.4}$$

for all $t \ge 0$*. If either*

(i) $p - 1 < q \le \frac{Q(p-1)}{Q-p}$ *and* $Q > p$*, or*

(ii) $0 \le q \le p - 1$ *and* $Q > 1$*,*

where Q *is defined as in* (1.2)*, then the problem*

$$-\mathrm{div}_G\Big(A(|\nabla_G u|)\nabla_G u\Big) \ge u^q \quad \text{in } \mathbb{R}^N = \mathbb{R}^{N_1} \times \mathbb{R}^{N_2} \tag{1.5}$$

has no nontrivial non-negative solutions of class $W^{1,p}_{\sigma,G,\mathrm{loc}}(\mathbb{R}^N)$*.*

Our approach in this paper is based on the test function method which was used in [2]. However, by the presence of the Grushin operator, we need to construct suitable test functions as well as establish nonlinear integral estimates corresponding to the Grushin operator.

2. Proof of Theorem 1.1

In this paper, letter C is used to denote a positive generic constant whose value does not matter, and to show that it depends on the associated parameters, we can add subscripts to C.

Proof in case when (i) holds. We consider the case $0 < p - 1 < q < \frac{Q(p-1)}{Q-p}$ and $Q > p$. Let $\varphi \in C_0^\infty(\mathbb{R}^N; [0,1))$ and $\sigma < 0$ (φ, σ will be chosen later). Without loss of generality, we can assume that $u > 0$ on $\mathbb{R}^N$ (otherwise, we replace u by $u_\varepsilon := u + \varepsilon$ and let $\varepsilon \downarrow 0$).

We will prove an important inequality as follows:

$$\int u^{q+\sigma}\varphi dz + \int |\nabla_G u|^p u^{\sigma-1}\varphi dz \le C \int \frac{|\nabla_G \varphi|^{pb_1}}{\varphi^{pb_1-1}} dz. \tag{2.1}$$

By multiplying with $u^\sigma\varphi$ both sides of (1.1) and integrating by parts, we get

$$\begin{aligned}\int u^{q+\sigma}\varphi dz &\le \sigma \int |\nabla_G u|^p u^{\sigma-1}\varphi dz + \int |\nabla_G u|^{p-2}(\nabla_G u, \nabla_G \varphi)u^\sigma dz \\ &\le \sigma \int |\nabla_G u|^p u^{\sigma-1}\varphi dz + \int |\nabla_G u|^{p-1}|\nabla_G \varphi|u^\sigma dz.\end{aligned}$$

By utilizing Young inequality with parameter $\varepsilon > 0$, we obtain

$$\int u^{q+\sigma}\varphi dz + |\sigma| \int |\nabla_G u|^p u^{\sigma-1}\varphi dz \leq \varepsilon \int |\nabla_G u|^p u^{\sigma-1}\varphi dz + C_\varepsilon \int u^{\sigma+p-1}\frac{|\nabla_G\varphi|^p}{\varphi^{p-1}}dz. \tag{2.2}$$

We put $\theta_\varepsilon = |\sigma| - \varepsilon$ to have

$$\int u^{q+\sigma}\varphi dz + \theta_\varepsilon \int |\nabla_G u|^p u^{\sigma-1}\varphi dz \leq C_\varepsilon \int u^{\sigma+p-1}\frac{|\nabla_G\varphi|^p}{\varphi^{p-1}}dz. \tag{2.3}$$

Since $0 < p-1 < q$, we can choose $\sigma < 0$ sufficiently close to 0 such that

$$a_1 = \frac{q+\sigma}{\sigma+p-1} > 1.$$

Applying again Young inequality with parameter $\varepsilon' > 0$ for the last term of (2.3), we obtain

$$\int u^{\sigma+p-1}\frac{|\nabla_G\varphi|^p}{\varphi^{p-1}}dz \leq \varepsilon' \int u^{q+\sigma}\varphi dz + C_{\varepsilon'} \int \frac{|\nabla_G\varphi|^{pb_1}}{\varphi^{pb_1-1}}dz, \tag{2.4}$$

where

$$\frac{1}{a_1} + \frac{1}{b_1} = 1.$$

By substituting (2.4) into (2.3), we get

$$(1-\varepsilon' C_\varepsilon)\int u^{q+\sigma}\varphi dz + \theta_\varepsilon \int |\nabla_G u|^p u^{\sigma-1}\varphi dz \leq C_\varepsilon C_{\varepsilon'} \int \frac{|\nabla_G\varphi|^{pb_1}}{\varphi^{pb_1-1}}dz.$$

We choose $\varepsilon, \varepsilon'$ sufficiently close to zero such that $\theta_\varepsilon > 0, 1-\varepsilon' C_\varepsilon > 0$, then we have

$$\int u^{q+\sigma}\varphi dz + \int |\nabla_G u|^p u^{\sigma-1}\varphi dz \leq C \int \frac{|\nabla_G\varphi|^{pb_1}}{\varphi^{pb_1-1}}dz.$$

The proof of (2.1) is completed. Now, we will prove the following inequality:

$$\int u^q\varphi dz \leq C\left(\int \frac{|\nabla_G\varphi|^{pb_1}}{\varphi^{pb_1-1}}dz\right)^{\frac{p-1}{p}+\frac{1}{pa_2}}\left(\int \frac{|\nabla_G\varphi|^{pb_2}}{\varphi^{pb_2-1}}dz\right)^{\frac{1}{pb_2}}. \tag{2.5}$$

By multiplying with φ both sides of (1.1) and integrating by parts, we obtain

$$\int u^q\varphi dz \leq \int |\nabla_G u|^{p-2}(\nabla_G u, \nabla_G\varphi)dz \leq \int |\nabla_G u|^{p-1}|\nabla_G\varphi|dz. \tag{2.6}$$

Applying Höder inequality, we have

$$\int |\nabla_G u|^{p-1}|\nabla_G\varphi|dz \leq \left(\int |\nabla_G u|^p u^{\sigma-1}\varphi dz\right)^{\frac{p-1}{p}}\left(\int u^{(1-\sigma)(p-1)}\frac{|\nabla_G\varphi|^p}{\varphi^{p-1}}dz\right)^{\frac{1}{p}}. \tag{2.7}$$

Substituting (2.7) into (2.6) to find

$$\int u^q \varphi dz \leq \left(\int |\nabla_G u|^p u^{\sigma-1} \varphi dz \right)^{\frac{p-1}{p}} \left(\int u^{(1-\sigma)(p-1)} \frac{|\nabla_G \varphi|^p}{\varphi^{p-1}} dz \right)^{\frac{1}{p}}. \tag{2.8}$$

Since $0 < p-1 < q$, we can choose $\sigma < 0$ sufficiently close to 0 such that

$$a_2 = \frac{q+\sigma}{(1-\sigma)(p-1)} > 1.$$

Applying again the Höder inequality, we obtain

$$\begin{aligned} \int u^{(1-\sigma)(p-1)} \frac{|\nabla_G \varphi|^p}{\varphi^{p-1}} dz &\leq \left(\int u^{(1-\sigma)(p-1)a_2} \varphi dz \right)^{\frac{1}{a_2}} \left(\int \frac{|\nabla_G \varphi|^{pb_2}}{\varphi^{pb_2-1}} dz \right)^{\frac{1}{b_2}} \\ &= \left(\int u^{q+\sigma} \varphi dz \right)^{\frac{1}{a_2}} \left(\int \frac{|\nabla_G \varphi|^{pb_2}}{\varphi^{pb_2-1}} dz \right)^{\frac{1}{b_2}}, \end{aligned}$$

where

$$\frac{1}{a_2} + \frac{1}{b_2} = 1.$$

Together with (2.8), we arrive at

$$\int u^q \varphi dz \leq \left(\int |\nabla_G u|^p u^{\sigma-1} \varphi dz \right)^{\frac{p-1}{p}} \left(\int u^{q+\sigma} \varphi dz \right)^{\frac{1}{pa_2}} \left(\int \frac{|\nabla_G \varphi|^{pb_2}}{\varphi^{pb_2-1}} dz \right)^{\frac{1}{pb_2}}.$$

Combining this with (2.1), we obtain

$$\int u^q \varphi dz \leq C \left(\int \frac{|\nabla_G \varphi|^{pb_1}}{\varphi^{pb_1-1}} dz \right)^{\frac{p-1}{p} + \frac{1}{pa_2}} \left(\int \frac{|\nabla_G \varphi|^{pb_2}}{\varphi^{pb_2-1}} dz \right)^{\frac{1}{pb_2}}.$$

The proof of (2.5) is finished.

Next, we choose the funtion φ. For $R > 0$, we put

$$\mathcal{B}_R = B_1(0, R) \times B_2(0, R^{\sigma+1}),$$

where $B_1(0,R), B_2(0,R^{\sigma+1})$ are the Euclidean balls in $\mathbb{R}^{N_1}, \mathbb{R}^{N_2}$, respectively. Similarly,

$$\mathcal{B}_{2R} = B_1(0, 2R) \times B_2(0, 2R^{\sigma+1}).$$

Let $\varphi_1, \varphi_2 \in C_c^\infty([0,+\infty)), 0 \leq \varphi_1, \varphi_2 \leq 1$ satisfy

$$\varphi_i(t) = \begin{cases} 1 & \text{in} \quad [0,1], \\ 0 & \text{in} \quad [2,+\infty). \end{cases}$$

Then, the funtions $\varphi_{1,R}, \varphi_{2,R}$ are defined by

$$\varphi_{1,R}(x) = \varphi_1\left(\frac{|x|}{R}\right), \varphi_{2,R}(y) = \varphi_2\left(\frac{|y|}{R^{\sigma+1}}\right),$$

and, for some constant $C > 0$, we obtain

$$|\nabla_x \varphi_{1,R}| \leq \frac{C}{R}, |\nabla_y \varphi_{2,R}| \leq \frac{C}{R^{\sigma+1}}.$$

For $z = (x, y) \in \mathbb{R}^N = \mathbb{R}^{N_1} \times \mathbb{R}^{N_2}$, where $x \in \mathbb{R}^{N_1}, y \in \mathbb{R}^{N_2}$, we define

$$\varphi_R(z) = \varphi_{1,R}(x)\varphi_{2,R}(y).$$

Then, $\varphi_R \in C_c^1(\mathbb{R}^N), 0 \leq \varphi_R \leq 1$ in $\mathbb{R}^N$ and we have

$$\begin{cases} \varphi_R = 1 & \text{in } \mathcal{B}_R, \\ \varphi_R = 0 & \text{in } \mathbb{R}^N \setminus \mathcal{B}_{2R}, \\ |\nabla_G \varphi_R| \leq \frac{C}{R} & \text{in } \mathcal{B}_{2R} \setminus \mathcal{B}_R. \end{cases}$$

By replacing φ with φ_R^λ, where λ is chosen sufficiently large such that $\lambda - pb_1 > 0, \lambda - pb_2 > 0$, it follows from (2.5) that

$$\begin{aligned} \int_{\mathcal{B}_R} u^q dz &\leq C\left(\int_{\mathcal{B}_{2R}\setminus\mathcal{B}_R} \varphi_R^{\lambda-pb_1}|\nabla_G\varphi_R|^{pb_1}dz\right)^{\frac{p-1}{p}+\frac{1}{pa_2}} \left(\int_{\mathcal{B}_{2R}\setminus\mathcal{B}_R} \varphi_R^{\lambda-pb_2}|\nabla_G\varphi_R|^{pb_2}dz\right)^{\frac{1}{pb_2}} \\ &\leq CR^m, \end{aligned} \tag{2.9}$$

where

$$m = (Q - pb_1)\left(\frac{p-1}{p} + \frac{1}{pa_2}\right) + (Q - pb_2)\frac{1}{pb_2}.$$

Since $a_1 = \frac{q+\sigma}{\sigma+p-1}$ and $a_2 = \frac{q+\sigma}{(1-\sigma)(p-1)}$, we can see that

$$b_1 = \frac{q+\sigma}{q-p+1}$$

and

$$b_2 = \frac{q+\sigma}{\sigma p + q - p + 1}.$$

Hence, we obtain

$$m = \frac{q(Q-p) - Q(p-1)}{q-p+1}. \tag{2.10}$$

Using the fact that $0 < p - 1 < q < \frac{Q(p-1)}{Q-p}$, we arrive $m < 0$. Let $R \to +\infty$ in (2.9), we obtain

$$\int_{\mathbb{R}^N} u^q dz = 0.$$

This contradicts with the assumption that $u > 0$.

Next, we consider the case $q = \frac{Q(p-1)}{Q-p}$. From (2.6), we have

$$\int u^q \varphi dz \le \int |\nabla_G u|^{p-2}(\nabla_G u, \nabla_G \varphi) dz \le \int |\nabla_G u|^{p-1} |\nabla_G \varphi| dz.$$

By putting $S(\nabla_G \varphi) = \mathrm{supp}(\nabla_G \varphi)$, we obtain

$$\int_{\mathcal{B}_R} u^q \varphi dz \le \int_{S(\nabla_G \varphi)} |\nabla_G u|^{p-1} |\nabla_G \varphi| dz. \tag{2.11}$$

Applying the Hölder inequality with $p_1 = \frac{p}{p-1}, q_1 = p$ for the right-hand side, we get

$$\int_{\mathcal{B}_R} u^q \varphi dz \le \Big(\int_{S(\nabla_G \varphi)} |\nabla_G u|^p u^{\sigma-1} \varphi dz \Big)^{\frac{p-1}{p}} \Big(\int_{S(\nabla_G \varphi)} u^{(1-\sigma)(p-1)} \frac{|\nabla_G \varphi|^p}{\varphi^{p-1}} dz \Big)^{\frac{1}{p}}. \tag{2.12}$$

Now, we can choose $\sigma < 0$ sufficiently close to zero and $q > p - 1 > 0$ such that

$$a_3 = \frac{q}{(1-\sigma)(p-1)} > 0.$$

Applying again Hölder inequality with $a_3, b_3 > 1 \Big(\frac{1}{a_3} + \frac{1}{b_3} = 1 \Big)$ for the last term of (2.12), we obtain

$$\int_{\mathcal{B}_R} u^q \varphi dz \le \left(\int_{S(\nabla_G \varphi)} |\nabla_G u|^p u^{\sigma-1} \varphi dz \right)^{\frac{p-1}{p}} \left(\int_{S(\nabla_G \varphi)} u^q \varphi dz \right)^{\frac{1}{pa_3}} \left(\int_{S(\nabla_G \varphi)} \frac{|\nabla_G \varphi|^{pb_3}}{\varphi^{pb_3-1}} dz \right)^{\frac{1}{pb_3}}. \tag{2.13}$$

Combining this with (2.1) and replacing φ by φ_R^λ, with λ is chosen sufficiently large, we get

$$\begin{aligned} \int_{\mathcal{B}_R} u^q dz &\le C \left(\int_{S(\nabla_G \varphi_R)} \frac{|\nabla_G \varphi_R|^{pb_1}}{\varphi_R^{pb_1-1}} dz \right)^{\frac{p-1}{p}} \left(\int_{S(\nabla_G \varphi_R)} \frac{|\nabla_G \varphi_R|^{pb_3}}{\varphi_R^{pb_3-1}} dz \right)^{\frac{1}{pb_3}} \\ &\quad \times \left(\int_{S(\nabla_G \varphi_R)} u^q \varphi_R dz \right)^{\frac{1}{pa_3}} \\ &\le C R^\theta \left(\int_{\mathcal{B}_{2R} \setminus \mathcal{B}_R} u^q dz \right)^{\frac{1}{pa_3}}, \end{aligned}$$

where

$$\theta = (Q - pb_1)\frac{p-1}{p} + (Q - pb_3)\frac{1}{pb_3}.$$

We deduce from $q = \frac{Q(p-1)}{Q-p}, b_1 = \frac{q+\sigma}{q-p+1}, \frac{1}{b_3} = 1 - \frac{1}{a_3} = 1 - \frac{(1-\sigma)(p-1)}{q}$ that $\theta = 0$. Therefore,

$$\int_{\mathcal{B}_R} u^q dz \le C \left(\int_{\mathcal{B}_{2R} \setminus \mathcal{B}_R} u^q dz \right)^{\frac{1}{pa_3}}. \tag{2.14}$$

By the assumption $q = \frac{Q(p-1)}{Q-p}$ and using (2.10), we get $m = 0$. Hence, from (2.9) we can deduce $\int_{\mathbb{R}^N} u^q dz < \infty$. Let $R \to +\infty$ in (2.14), we arrive at

$$\int_{\mathbb{R}^N} u^q dz = 0.$$

We deduce a contradiction. This completes the proof of item (i).

Proof the case when (ii) holds. Let us consider the case $0 \le q < p-1$. From (2.3), by choosing $\varepsilon > 0$ small enough such that $\theta_\varepsilon > 0$, we have

$$\int u^{q+\sigma} \varphi dz \le C \int u^{\sigma+p-1} \frac{|\nabla_G \varphi|^p}{\varphi^{p-1}} dz. \tag{2.15}$$

Let $\sigma = q(Q-p) - Q(p-1)$. Since $0 \le q < p-1$ and $Q > 1$, we get $\sigma < 0$. Beside, we have $\sigma < 1-p < -q$ and $a_1 = \frac{q+\sigma}{\sigma+p-1} > 1$. Applying the Höder inequality with $a_1 = \frac{q+\sigma}{\sigma+p-1}$ and b_1 such that $\frac{1}{a_1} + \frac{1}{b_1} = 1$ for the right-hand side of (2.15), we obtain

$$\int u^{q+\sigma} \varphi dz \le C \Big(\int u^{q+\sigma} \varphi dz \Big)^{\frac{1}{a_1}} \Big(\int \frac{|\nabla_G \varphi|^{pb_1}}{\varphi^{pb_1-1}} dz \Big)^{\frac{1}{b_1}}.$$

Take $\varphi = \varphi_R^\lambda$, we arrive at

$$\int u^{q+\sigma} \varphi_R dz \le C \int \frac{|\nabla_G \varphi_R|^{pb_1}}{\varphi_R^{pb_1-1}} dz \le C R^{Q-pb_1}.$$

Hence,

$$\int_{\mathcal{B}_R} u^{q+\sigma} dz \le C R^{Q-pb_1}. \tag{2.16}$$

Since $Q - pb_1 = \frac{Q(q-p+1) - p(q+\sigma)}{q-p+1} < 0$, let $R \to +\infty$ in (2.16), we get

$$\int_{\mathbb{R}^N} u^{q+\sigma} dz = 0.$$

This contradicts with the assumption that $u > 0$. The proof of item (ii) is completed.

Next, considering the case $q = p-1$. By choosing $\sigma = 1-p < 0$, we have $q + \sigma = 0$. Therefore, from (2.15) and replacing φ by φ_R^λ with λ is chosen sufficiently large, we get

$$\int_{\mathbb{R}^N} \varphi_R dz \le C \int \frac{|\nabla_G \varphi_R|^{pb_1}}{\varphi_R^{pb_1-1}} dz.$$

Hence,

$$R^Q \le C R^{Q-pb_1},$$

i.e.,

$$1 \le C R^{-pb_1}.$$

Let $R \to +\infty$ we reach a contradiction. Therefore, we get the conclusion of the theorem. □

3. Proof of Theorem 1.2

Proof in case when (i) holds. It is similar to the proof of Theorem 1.1, we can assume that $u > 0$ on $\mathbb{R}^N$. Suppose that $\varphi \in C_0^\infty(\mathbb{R}^N; [0,1)])$ and $\sigma < 0$ which will be chosen later. Multiplying $u^\sigma \varphi$ on both sides of (1.5) and integrating by parts, leads to

$$\int u^{q+\sigma}\varphi dz \le \sigma \int A\big(|\nabla_G u|\big)|\nabla_G u|^2 u^{\sigma-1}\varphi dz + \int A\big(|\nabla_G u|\big)(\nabla_G u, \nabla_G \varphi) u^\sigma dz.$$

Therefore,

$$\int u^{q+\sigma}\varphi dz + |\sigma| \int A\big(|\nabla_G u|\big)|\nabla_G u|^2 u^{\sigma-1}\varphi dz \le \int A\big(|\nabla_G u|\big)|\nabla_G u||\nabla_G \varphi| u^\sigma dz.$$

From (1.4), we obtain

$$\int u^{q+\sigma}\varphi dz + c_1|\sigma| \int |\nabla_G u|^p u^{\sigma-1}\varphi dz \le c_2 \int |\nabla_G u|^{p-1}|\nabla_G \varphi| u^\sigma dz.$$

Applying Young inequality with parameter $\varepsilon > 0$ for the right-hand side of the above inequality, we obtain

$$\int u^{q+\sigma}\varphi dz + c_1|\sigma| \int |\nabla_G u|^p u^{\sigma-1}\varphi dz \le c_2\varepsilon \int |\nabla_G u|^p u^{\sigma-1}\varphi dz + c_2 C_\varepsilon \int u^{\sigma+p-1}\frac{|\nabla_G \varphi|^p}{\varphi^{p-1}} dz. \tag{3.1}$$

We put $\theta_\varepsilon = c_1|\sigma| - c_2\varepsilon$ to have

$$\int u^{q+\sigma}\varphi dz + \theta_\varepsilon \int |\nabla_G u|^p u^{\sigma-1}\varphi dz \le c_2 C_\varepsilon \int u^{\sigma+p-1}\frac{|\nabla_G \varphi|^p}{\varphi^{p-1}} dz. \tag{3.2}$$

Since $0 < p-1 < q$, we can choose $\sigma < 0$ sufficiently close to 0 such that

$$a_1 = \frac{q+\sigma}{\sigma+p-1} > 1.$$

Applying again the Young inequality with parameter $\varepsilon^{'} > 0$ for the last term of (3.2), we obtain

$$\int u^{\sigma+p-1}\frac{|\nabla_G \varphi|^p}{\varphi^{p-1}} dz \le \varepsilon^{'} \int u^{q+\sigma}\varphi dz + C_{\varepsilon'} \int \frac{|\nabla_G \varphi|^{pb_1}}{\varphi^{pb_1-1}} dz, \tag{3.3}$$

where

$$\frac{1}{a_1} + \frac{1}{b_1} = 1.$$

By substituting (3.3) into (3.2), we obtain

$$(1 - c_2\varepsilon^{'} C_\varepsilon) \int u^{q+\sigma}\varphi dz + \theta_\varepsilon \int |\nabla_G u|^p u^{\sigma-1}\varphi dz \le c_2 C_\varepsilon C_{\varepsilon'} \int \frac{|\nabla_G \varphi|^{pb_1}}{\varphi^{pb_1-1}} dz, \tag{3.4}$$

By choosing $\varepsilon, \varepsilon'$ sufficiently close to zero such that $\theta_\varepsilon > 0, 1 - c_2\varepsilon' C_\varepsilon > 0$, we have

$$\int u^{q+\sigma}\varphi dz + \int |\nabla_G u|^p u^{\sigma-1}\varphi dz \le C \int \frac{|\nabla_G \varphi|^{pb_1}}{\varphi^{pb_1-1}} dz. \tag{3.5}$$

Multiplying φ on both sides of (1.5) and integrating by parts, leads to

$$\int u^q \varphi dz \le \int A\big(|\nabla_G u|\big)|\nabla_G u||\nabla_G \varphi| dz \le c_2 \int |\nabla_G u|^{p-1}|\nabla_G \varphi| dz.$$

By applying Höder inequality, we have

$$\int u^q \varphi dz \le c_2 \bigg(\int |\nabla_G u|^p u^{\sigma-1}\varphi dz\bigg)^{\frac{p-1}{p}} \bigg(\int u^{(1-\sigma)(p-1)} \frac{|\nabla_G \varphi|^p}{\varphi^{p-1}} dz\bigg)^{\frac{1}{p}}. \tag{3.6}$$

Since $0 < p - 1 < q$, we can choose $\sigma < 0$ sufficiently close to 0 such that

$$a_2 = \frac{q+\sigma}{(1-\sigma)(p-1)} > 1.$$

Applying again the Höder inequality, we obtain

$$\begin{aligned}\int u^{(1-\sigma)(p-1)} \frac{|\nabla_G \varphi|^p}{\varphi^{p-1}} dz &\le \bigg(\int u^{(1-\sigma)(p-1)a_2}\varphi dz\bigg)^{\frac{1}{a_2}} \bigg(\int \frac{|\nabla_G \varphi|^{pb_2}}{\varphi^{pb_2-1}} dz\bigg)^{\frac{1}{b_2}} \\ &= \bigg(\int u^{q+\sigma}\varphi dz\bigg)^{\frac{1}{a_2}} \bigg(\int \frac{|\nabla_G \varphi|^{pb_2}}{\varphi^{pb_2-1}} dz\bigg)^{\frac{1}{b_2}},\end{aligned}$$

where

$$\frac{1}{a_2} + \frac{1}{b_2} = 1.$$

Substituting into (3.6), we arrive at

$$\int u^q \varphi dz \le c_2 \bigg(\int |\nabla_G u|^p u^{\sigma-1}\varphi dz\bigg)^{\frac{p-1}{p}} \bigg(\int u^{q+\sigma}\varphi dz\bigg)^{\frac{1}{pa_2}} \bigg(\int \frac{|\nabla_G \varphi|^{pb_2}}{\varphi^{pb_2-1}} dz\bigg)^{\frac{1}{pb_2}}.$$

Combining this with (3.5), we obtain

$$\begin{aligned}\int u^q \varphi dz &\le C \bigg(\int \frac{|\nabla_G \varphi|^{pb_1}}{\varphi^{pb_1-1}} dz\bigg)^{\frac{p-1}{p}} \bigg(\int \frac{|\nabla_G \varphi|^{pb_1}}{\varphi^{pb_1-1}} dz\bigg)^{\frac{1}{pa_2}} \bigg(\int \frac{|\nabla_G \varphi|^{pb_2}}{\varphi^{pb_2-1}} dz\bigg)^{\frac{1}{pb_2}} \\ &= C \bigg(\int \frac{|\nabla_G \varphi|^{pb_1}}{\varphi^{pb_1-1}} dz\bigg)^{\frac{p-1}{p} + \frac{1}{pa_2}} \bigg(\int \frac{|\nabla_G \varphi|^{pb_2}}{\varphi^{pb_2-1}} dz\bigg)^{\frac{1}{pb_2}}.\end{aligned} \tag{3.7}$$

The rest of the proof is similar to that of Theorem 1.1 and so will be omitted. □

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