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STRONG-COUPLING PERTURBATION THEORY IN A SIMPLE AND EXACTLY SOLVABLE MODEL

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Abstract. We study strong-coupling perturbation theory in a simple and exactly solvable model. The divergence of the perturbative series is effectively addressed using the technique of continued fractions. The results obtained from perturbation theory show good agreement with the exact solutions. The calculations are straightforward, making them suitable for teaching quantum mechanics or other theoretical physics courses, as well as for application in more complex research studies.

Keywords: strong-coupling, perturbation theory, divergence.

1. Introduction

Perturbation theory [1] is a powerful and versatile analytical technique that finds application across numerous disciplines, including science and engineering [2]. In physics, however, it is typically introduced within the context of quantum problems, where it is closely linked to concepts such as interactions, actions, and functionals, particularly in areas like ϕ^{2N} theory [3], [4], Quantum Electrodynamics (QED) [5], [6], Quantum Chromodynamics (QCD) [7], Fermion systems [8], and Yang-Mills theories [9]. This specialized focus can sometimes create the misconception that perturbative methods are confined to these particular frameworks.

In contrast, in purely mathematical settings [10] such as solving algebraic or differential equations, the concept of "interaction" has no direct relevance. Identifying an unperturbed problem and determining perturbative corrections, as commonly done in physics, may not be necessary [11]. In these mathematical contexts, terms can be interpreted without attaching any physical significance, allowing for a more abstract and generalized application of the perturbation technique [12], [13].

This specialized focus on quantum problems [14] can sometimes hinder physics students from applying perturbation theory to new, unrelated problems. Thus, it would be beneficial to teach perturbation theory from a broader, more generalized perspective, employing simpler examples to illustrate the fundamental principles. This approach would better equip students to apply the method across diverse situations [15].

At its core, perturbation theory simplifies a complex problem by approximating it through an infinite series of simpler, analytically solvable problems [15]. The method relies on a small parameter, known as the perturbation parameter. When this parameter is zero, we recover the unperturbed problem; when it is one, we solve the actual problem of interest. The solution is expressed as a series of successive corrections. The primary challenge lies in summing this series, as it may converge slowly or even diverge [16], [17]. In such cases, standard perturbation theory becomes inadequate, necessitating alternative approaches [18], [19].

In the weak-coupling limit, where perturbation theory is most effective, the series converges rapidly, yielding highly accurate predictions, especially with higher-order corrections. In some instances, transformations like the Shank transformation can achieve high precision even with lower-order terms [20], [21]. However, in the strong-coupling regime, the perturbative series often fails to converge, prompting physicists to employ more advanced non-perturbative techniques or modify the perturbative approach to manage divergences [16]-[19].

Some advances in the study of divergent series suggest that acquiring skills to handle them should be an integral part of the graduate curriculum in theoretical physics [15], [16]. Yet, the theory of divergent series remains largely unfamiliar to most physicists, as it is typically absent from standard calculus courses that date back to the mid-19th century, a period when divergent series were nearly excluded from mathematical study. For a brief overview of the mathematical theory behind divergent series, readers may refer to sources [15], [16], and [20].

Addressing divergences in strong-coupling scenarios requires specialized methods to render perturbative series practically useful. A common approach involves regularization and renormalization, systematically canceling or absorbing divergent terms into redefined parameters to stabilize the series [18]. Furthermore, resummation techniques such as continued fractions, Borel resummation, or Padé approximants are frequently used to reorganize series expansions and enhance their convergence properties [13], [15], [20]. These strategies extend the utility of perturbation theory even in cases where conventional series expansions yield unreliable results, thus enabling approximate solutions when standard methods falter. In such contexts, the refined methodology is known as Strong-Coupling Perturbation Theory (SCPT), emphasizing its utility in tackling systems where the typical weak-coupling assumptions do not apply. Despite its potential, SCPT is often introduced in the academic literature within the context of quantum mechanics, accompanied by complex concepts like Hamiltonians, wave functions, and Green’s functions [11]. This can obscure the underlying principles of SCPT, making it seem more complex than it truly is, especially for students who encounter intricate mathematical calculations tied to specific models rather than the core perturbative techniques.

In this paper, we aim to demystify SCPT by applying it in a simpler, non-quantum context using an exactly solvable model. This approach allows readers to grasp the essence of SCPT more intuitively and directly compare its results with exact solutions.

By simplifying the context, we seek to provide clearer insights into the effectiveness of the perturbative method, making it more accessible and applicable to a broader range of problems.

In the following sections, we introduce a straightforward physical problem that can be solved exactly, illustrating the application of perturbation theory in both weak-coupling and strong-coupling limits. The calculations are presented step-by-step, with detailed explanations to ensure that readers can apply these methods in various contexts. We conclude with a discussion of the findings and recommendations for further exploration.

2. Content

2.1. A simple and exactly solvable model

In order to meet the objectives set forth in the introduction, we have selected a simple mechanics problem from the high school physics curriculum. This problem involves motion in a uniform gravitational field without friction. The specific problem is as follows: A small stone is thrown vertically downward from a height h with an initial velocity v_0. Determine the time it takes for the stone to hit the ground, assuming a constant gravitational acceleration g. To further simplify the calculations, let's use a specific set of values: h = 5m, v_0 = 5m/s, g = 10 m/s^2. All units for the quantities are given in the SI system.

Choose the z-axis pointing upwards, with the origin at ground level and the time origin set at the moment the object is thrown. The position of the object at time t > 0 is given by $z = h - v_0 t - \frac{1}{2} g t^2$. When the object hits the ground, we have z = 0. Thus t is the positive solution of the following equation:

$$0 = h - v_0 t - \frac{1}{2} g t^2. \qquad (1)$$

With these given values of the height h, initial speed v_0 and gravitational acceleration g, one has

$$t^2 + t - 1 = 0, \qquad (2)$$

where the exact solution t = $\frac{-1+\sqrt{5}}{2} \approx 0,6180339887$ s.

With the selected problem, we have easily obtained the exact solution. Next, we will sequentially apply perturbation theory in both the weak-coupling and strong-coupling regimes.

2.2. Weak coupling perturbation theory

To solve the problem in the weak-coupling regime, we will treat the initial velocity as a perturbation, introducing the perturbation parameter ε into the first-order term. The perturbative problem in the weak-coupling limit thus becomes:

$$t^2 + \varepsilon t - 1 = 0 \qquad (3)$$

Step 1: With $\varepsilon = 0$ The unperturbed problem is the free-fall problem. Substituting $\varepsilon = 0$ into (3), we get the positive solution t = 1.

Step 2: Find the solution of the equation in the form $ANS(\varepsilon)=t(\varepsilon)=\sum_{n=0}^{\infty}a_n\varepsilon^n$. Substitute this into equation (3), and we get $\left(\sum_{n=0}^{\infty}a_n\varepsilon^n\right)^2+\varepsilon\left(\sum_{n=0}^{\infty}a_n\varepsilon^n\right)-1=0$. For the zeroth-order approximation, we have $a_0=1$.

For the first-order approximation, we have $ANS(\varepsilon)=1+a_1\varepsilon$. Substitute this into (3), and we get $(1+a_1\varepsilon)^2+\varepsilon(1+a_1\varepsilon)-1=0\Rightarrow 2a_1\varepsilon+\varepsilon=0$. Thus one obtains $a_1=-\frac{1}{2}$.

Thus, the solution to the problem is $ANS(\varepsilon)=1-\frac{1}{2}\varepsilon$. The meaning of this first correction is that it represents the time change due to the difference between the initial velocity v_0 of the perturbed problem and the initial velocity of the unperturbed problem. Since the initial velocity is directed downward, the first-order correction is negative.

Step 3: Substitute $\varepsilon=1$ and sum the series to find the solution to the problem. Using only the first-order term, we get $ANS(1)=1-\frac{1}{2}=0{,}5$. The result shows an error of approximately 19.1%. This error is not acceptable, so we proceed to find the next terms.

For the second-order approximation, we have $ANS(\varepsilon)=1-\frac{1}{2}\varepsilon+a_2\varepsilon^2$. Substitute this into equation (3), and we get $\left(1-\frac{1}{2}\varepsilon+a_2\varepsilon^2\right)^2+\varepsilon\left(1-\frac{1}{2}\varepsilon+a_2\varepsilon^2\right)-1=0$. One finds $a_2=\frac{1}{8}$.

The solution to the problem is $ANS(\varepsilon)=1-\frac{1}{2}\varepsilon+\frac{1}{8}\varepsilon^2$.

Substituting $\varepsilon=1$ we get $ANS(1)=1-\frac{1}{2}+\frac{1}{8}=0{,}625$. The result shows an error of approximately 1.13%. Thus, by considering up to the second order, the error has decreased significantly and is now acceptable. We can continue the above process to calculate higher-order corrections to any desired order. For comparison later, we have calculated up to the sixth order; the solution is $ANS(\varepsilon)=1-\frac{1}{2}\varepsilon+\frac{1}{8}\varepsilon^2-\frac{1}{128}\varepsilon^4+\frac{1}{1024}\varepsilon^6$.

Substituting $\varepsilon=1$ we get $ANS(1)=1-\frac{1}{2}+\frac{1}{8}-\frac{1}{128}+\frac{1}{1024}=0{,}6181640625$. We see that the result is correct to three decimal places, with an error of approximately 0.013%.

Thus, the perturbation theory in the weak-coupling limit converges quite rapidly. The convergence rate can be further improved using modern techniques, such as Shanks transformation [16], [20], [21]. The Shanks transformation $S(S_N)$ of the N-th partial sum S_N of the series is given by

$$S(S_N)=\frac{S_N^2-S_{N-1}S_{N+1}}{2S_N-S_{N+1}-S_{N-1}},$$

where the resulting values, $S(S_N)$, can be further transformed iteratively using the same method by the Shanks formula, yielding S(S(SN)), and so forth. One obtains Table 1.

Table 1. Shanks transformation for the weak coupling problem

S_N	$S(S_N)$	$S(S(S_N))$
1,000 000 000		
0,500 000 000	0,600 000 000	
0,625 000 000	0,617 647 058	0,618 065 235
0,617 187 500	0,618 055 555	
0,618 164 062		

We observe that using the standard summation method, the result is accurate to three decimal places. However, by applying the Shanks transformation, the result improves to four decimal places. This demonstrates that the Shanks transformation significantly enhances the convergence rate of the series without needing to calculate higher-order perturbations. However, it is important to note that the Shanks transformation can only accelerate the convergence of a convergent series; it cannot be applied to a divergent series. In practice, we often encounter situations where the perturbation series does not converge. To investigate how to handle divergent series, we will consider the strong-coupling limit in the next section.

2.3. Strong coupling perturbation theory

To solve the problem in the strong-coupling regime, we will treat the gravitational acceleration g as a perturbation, introducing the perturbation parameter ε into the second-order term. The perturbative problem in the strong-coupling limit thus becomes

$$\varepsilon t^2 + t - 1 = 0 \tag{4}$$

Step 1: With $\varepsilon = 0$ The non-perturbation problem is the problem of uniform motion, where the time of motion can be easily determined as t =1.

Step 2: Find the solution of the equation in the $ANS(\varepsilon) = t(\varepsilon) = \sum_{n=0}^{\infty} a_n \varepsilon^n$. Substitute this into equation (4), and we get $\varepsilon\left(\sum_{n=0}^{\infty} a_n \varepsilon^n\right)^2 + \left(\sum_{n=0}^{\infty} a_n \varepsilon^n\right) - 1 = 0$. For the zeroth-order approximation, we have $a_0 = 1$.

For the first-order approximation, we have $ANS(\varepsilon) = 1 + a_1\varepsilon$. Substitute this into (4), and we get $\varepsilon(1 + a_1\varepsilon)^2 + (1 + a_1\varepsilon) - 1 = 0$. One obtains $a_1 = -1$. The calculation is performed in the same way as in the previous section, so we will not go into detail here. The only difference is that, in this case, the perturbation series does not converge.

Step 3: We can continue the above process to calculate higher-order corrections to any desired order. For the purpose of comparison later, we have calculated up to the fifth order; the solution is

$$ANS(\varepsilon) = 1 - \varepsilon + 2\varepsilon^2 - 5\varepsilon^3 + 14\varepsilon^4 - 42\varepsilon^5. \tag{5}$$

The perturbation series in the strong coupling limit of the problem under consideration is clearly a divergent series with a zero radius of convergence. Typically, one would conclude that the perturbation method does not yield reliable results and switch to non-perturbative methods. However, we will handle this divergent series using continued fraction techniques. We will find a continued fraction whose Taylor expansion up to the fifth order matches the first five terms of the divergent series in equation (5). Then, we will compute the result of the series from this continued fraction by setting $\varepsilon = 1$. We found that to the fifth order,

$$\mathrm{ANS}(\varepsilon) = \sum a_n \varepsilon^n = \cfrac{b_0}{1 + b_1 \cfrac{\varepsilon}{1 + b_2 \cfrac{\varepsilon}{1 + b_3 \cfrac{\varepsilon}{1 + b_4 \cfrac{\varepsilon}{1 + b_5 \varepsilon}}}}} \tag{6}$$

with $b_0 = b_1 = b_2 = b_4 = b_5 = 1$. Now we can set $\varepsilon = 1$ To calculate the values of the obtained fraction, the solution turns out to be $\mathrm{ANS}(1) = 0,6153846154$ with the error of about 0,43%.

Here, we see that after applying the continued fraction technique, we can find a meaningful solution to the problem in the strong-coupling limit. Although the error may be larger than in the weak-coupling case, obtaining a finite value instead of a meaningless result from a divergent series is already an improvement. Typically, researchers would be satisfied with this and stop there. However, we can take it a step further by applying the Shanks transformation to the continued fraction obtained after processing the perturbation series. The N-th partial sum of the perturbation series, of order N, is computed using the corresponding N-th order continued fraction. The results are quite impressive; by applying the Shanks transformation after the continued fraction, we achieve solutions with much greater accuracy, as shown in Table 2.

Table 2. Shanks transformation for the strong coupling problem

S_N	$S(S_N)$	$S(S(S_N))$	$S(S(S(S_N)))$
1,000 000 0000			
0,500 000 0000	0,625 000 000		
0,666 666 6670	0,619 047 619	0,618 034 448	
0,600 000 0000	0,618 181 818	0,618 033 999	0,618 033 9887
0,625 000 0000	0,618 055 556	0,618 033 989	
0,615 384 6154	0,618 037 135		
0,619 047 6190			

We can see that applying the Shanks transformation yields remarkable results in this case. Using the standard summation method, the result is accurate to only three decimal places. However, after applying the Shanks transformation, the result is accurate to ten decimal places. Thus, for a divergent series, using continued fractions and the

Shanks transformation can yield high accuracy more rapidly than repeatedly summing a convergent series.

3. Conclusions

In this study, we explored the application of Strong-Coupling Perturbation Theory (SCPT) in a simple, exactly solvable model. By examining both weak-coupling and strong-coupling limits, we showed how perturbation series can provide useful approximations to the exact solution. While the weak-coupling approach produced highly accurate results with fast convergence, the strong-coupling scenario highlighted the challenges posed by divergent series.

To address this issue, we used continued fraction techniques to handle the divergent series in the strong-coupling limit. This approach allowed us to obtain meaningful solutions despite the series divergence. Furthermore, by applying the Shanks transformation to the continued fraction, we achieved significantly improved accuracy, with results that were remarkably close to the exact solution.

Our findings demonstrate the effectiveness of continued fractions and the Shanks transformation in handling divergent perturbation series, providing accurate results where standard methods fail. This approach offers a powerful tool for addressing complex problems in quantum mechanics and other fields of theoretical physics involving strong-coupling effects. The calculations can also be applied to problems like the one in reference [22].

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