RANK OF THE DERIVATIVE OF THE PROJECTION TO SYMMETRIZED POLYDISC

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Abstract. The projection, also called the symmetrization mapping, from spectral ball to symmetrized polydisc is closely related to the spectral Nevanlinna-Pick interpolation problem. We prove that the rank of the derivative of the projection from the spectral unit ball to the symmetrized polydisc is equal to the degree of the minimal polynomial of the matrix at which we take the derivative. Therefore, it explains why the corresponding lifting problem is easier when the matrix base-point is cyclic since it is a regular point of the symmetrization mapping in the differential sense.
Keywords: Nevanlinna-Pick, interpolation, symmetrized polydisc.

1. Introduction

In this note, we are interested in a special mapping in the spectral Nevanlinna-Pick problem which is called the symmetrization mapping. First we present some notations.

Denote by $\mathbb{C}^{n,n}$ the set of complex square matrices of size n, where n is a positive integer. For each matrix $M \in \mathbb{C}^{n,n}$, the characteristic polynomial of M is defined as

$$P_M(t) = \det(tI - M) = \sum_{j=0}^{n} (-1)^j \sigma_j(M) t^{n-j}$$

where $\sigma_0(M) = 1$ by convention, $\sigma_j(M)$ are the coefficients of the characteristic polynomial $\det(tI - M)$ and I is the identity matrix. The $\sigma_j(M)'s$ are in fact the elementary symmetric functions of the eigenvalues of M.

Put $\pi(M) = (\sigma_1(M), \sigma_2(M), \ldots, \sigma_n(M))$ so we get a mapping $\pi \colon \mathbb{C}^{n,n} \to \mathbb{C}^n$ which is called the *symmetrization mapping*.

Next we put

$$\Omega_n = \{M \in \mathbb{C}^{n,n} \ : \ r(M) < 1\}$$

where $r(M)$ is the spectral radius of M. We call Ω_n the *spectral ball* and its image

$$\mathbb{G}_n = \pi(\Omega_n)$$

the *symmetrized polydisc*. This object is first introduced by Agler J and Young N [1] to study spectral Nevanlinna-Pick problem, i.e., given $\alpha_1, \alpha_2, \ldots, \alpha_m \in \mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ and $A_1, A_2, \ldots, A_m \in \Omega_n$, find conditions such that there exists a holomorphic function $\Phi \colon \mathbb{D} \to \Omega_n$ with $\Phi(\alpha_j) = A_j$ for $1 \leq j \leq m$.

Unfortunately, this problem is extremely difficult since Ω_n has an unfriendly geometry in comparison to the homogeneity of the disc or the unit ball in the classical Nevanlinna-Pick interpolation problem. Therefore, Agler J and Young N propose a new idea that instead of studying Ω_n, they project Ω_n onto $\mathbb{G}_n$ by symmetrization map and this object is a bounded taut domain, so theoretically it is easier to deal with. Since then, a lot of papers have appeared to study the symmetrized polydisc and give interesting information.

In the inverse direction, we are interested in the lifting problem, i.e., if we can realize interpolation in $\mathbb{G}_n$, can we lift it to Ω_n? A partial question has been resolved in [2].

In this note, we are interested in the symmetrization mapping $\pi \colon \Omega_n \to \mathbb{G}_n$ and prove that the rank of the derivative $\pi'(B)$ is equal to the degree of the polynomial of B for any matrix $B \in \Omega_n$. This result explains in part why the lifting problem is always realizable in [3].

2. Content

2.1. Notations and local lifting problem

For $\varepsilon > 0$, we denote by

$$\mathbb{D}_\varepsilon = \{z \in \mathbb{C} : |z| < \varepsilon\}$$

the disc of radius ε.

For a vector $v = (v_1, v_2, \ldots, v_n) \in \mathbb{C}^n$, put

$$P_{[v]}(t) = \sum_{j=0}^{n} (-1)^j v_j t^{n-j}$$

where $v_0 = 1$. This notation ensures that

$$P_{\pi(M)}(t) = \det(tI - M)$$

for $M \in \mathbb{C}^{n,n}$.

[local lifting problem] Given a holomorphic mapping $\varphi \colon \mathbb{D}_\varepsilon \to \mathbb{G}_n$ and $B \in \Omega_n$ such that $\varphi(0) = \pi(B)$. The *local lifting problem* for φ asks whether there exists a holomorphic mapping $\Phi \colon \mathbb{D}_{\varepsilon'} \to \Omega_n$ such that $0 < \varepsilon' < \varepsilon$, $\Phi(0) = B$ and $\pi \circ \Phi = \varphi$.

2.2. Main result

We will prove the following result in the remaining sections.

Consider the symmetrization mapping $\pi \colon \Omega_n \to \mathbb{G}_n$. Then for any $B \in \Omega_n$, we have that the rank of the derivative of π at B is equal to the degree of the minimal polynomial of B.

We recall here the definition of the minimal polynomial of a square matrix $B \in \mathbb{C}^{n,n}$.

Let $B \in \mathbb{C}^{n,n}$ be a square matrix and $P(t)$ be a complex polynomial in t. We say that $P(t)$ is the minimal polynomial of B if $P(t)$ is of the smallest positive degree such that $P(B) = 0$.

To prove this result, we have to make use of Jordan's normal form which is recalled in the next section.

2.3. Jordan normal form and notations

By taking conjugation if necessary, we can suppose B is in the Jordan normal form, i.e., B is in the form of a block matrix where each block in the diagonal is an elementary Jordan block corresponding to an eigenvalue. We always gather all the elementary Jordan blocks corresponding to the same eigenvalue into a big Jordan block.

It means that $B = \begin{bmatrix} B_1 & & & \\ & B_2 & & \\ & & \ddots & \\ & & & B_m \end{bmatrix}$ where $B_1, B_2, \ldots, B_m$ are big Jordan

blocks corresponding to distinct eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_m$. Temporarily, we fix an eigenvalue λ_k and its block B_k and denote by (λ, B) (i.e. without "k").

So B here has only one eigenvalue, which is λ, and B is of size m_λ, the algebraic multiplicity of λ.

We suppose B is of the form

$$\begin{bmatrix} \lambda & b_{1,2} & 0 & & \\ & \lambda & b_{2,3} & & \\ & & \ddots & & \\ & & & \lambda & b_{m_\lambda-1,m_\lambda} \\ & & & & \lambda \end{bmatrix}$$

where the entries $b_{j-1,j} \in \{0, 1\}$ and all the blank entries are equal to zero.

Then we follow [2] and put

$$F_0^\lambda = \{j \ : \ b_{j-1,j} = 0\} = \{1 = b_1 < b_2 < \ldots < b_s\}$$

and suppose

$$b_2 - b_1 \leq b_3 - b_2 \leq \ldots \leq b_{s+1} - b_s$$

where $b_{s+1} = m_\lambda + 1$. This means we arrange the elementary Jordan blocks in B in an increasing order of sizes from the left to the right.

Next put $d_i^\lambda = 1 + \#(F_0^\lambda \cap [m_{\lambda-i+2..m_\lambda}])$.

If $\Phi \colon \mathbb{D}_\varepsilon \to \Omega_n$ is a holomorphic mapping with $\Phi(0) = B$, and if $\varphi = \pi \circ \Phi$, then we have

$$P_{[\varphi(\zeta)]}^{(k)}(\lambda) = O(\zeta^{d_{m_\lambda-k}^\lambda}) \text{ for } 0 \leq k \leq m_\lambda - 1 \tag{2.1}$$

where O is the big O Landau notation and $P_{[\varphi(\zeta)]}^{(k)}$ is the $k-$th derivative of the polynomial with respect to its own variable (so ζ is the variable of φ). This computation follows from the fact that

$$P_{[\varphi(\zeta)]}^{(k)}(\lambda) = k!\sigma_k(\lambda I - \Phi(\zeta)).$$

The estimate in (2.1) was first stated and proved in [4].

2.4. Proof of the main result

Now we present the proof of Theorem 2.2.. As before, B can be replaced by any similar matrix, so we can suppose B is in the Jordan normal form. Denote by m the degree of the minimal polynomial of B. For each eigenvalue λ (without counting multiplicities) of B, we denote by s_λ the size of the biggest elementary Jordan block corresponding to λ appearing in B. Then we have

$$m = \sum_{\lambda \in \mathrm{Sp}(B)} s_\lambda$$

where $Sp(B)$ is the spectrum of B. For the proof of this result, cf [5, Chapter 2, Corollary 2.3.12, page 31] or for the statement of the result (without proof), cf the Vietnamese textbook [6, Chapter 5, Proposition 5.5.4, page 139].

To prove the equality $\mathrm{rank}(\pi'(B)) = m$, we prove two inequalities and divide the proof into two parts.

2.4.1. The rank is less than or equal to the degree

First, we prove $\mathrm{rank}(\pi'(B)) \leq m$. To do it, we prove that the image of $\pi'(B)$ in $\mathbb{C}^n$ is a vector subspace of dimension less than $\leq m$. We realize this by showing that this space is the solution set of a system of homogeneous linear equations with explicit coefficients as follows.

Put $\Phi(\zeta) = B + \zeta M$ for any $M \in \mathbb{C}^{n,n}$ and $\varphi = \pi \circ \Phi$. Then we have

$$\varphi'(0) = (\pi \circ \Phi)'(0) = \pi'(B)M.$$

Therefore, $\varphi'(0)$ can be any vector in the image of $\pi'(B)$.

By (2.1), we have

$$P^{(k)}_{[\varphi(\zeta)]}(\lambda) = O(\zeta^{d^\lambda_{m_\lambda - k}}) \text{ for } 0 \le k \le m_\lambda - 1.$$

Recall that $P_{[\varphi(\zeta)]}(t) = \sum_{j=0}^n (-1)^j \varphi_j(\zeta) t^{n-j}$ with convention $\varphi_0(\zeta) = 1$. Put $v(t) = (-t^{n-1}, t^{n-2}, \ldots, (-1)^{n-1}t, (-1)^n)$. For two complex vectors $u = (u_1, u_2, \ldots, u_n)$ and $v = (v_1, v_2, \ldots, v_n) \in \mathbb{C}^n$, put

$$u \cdot v = \sum_{j=1}^n u_j v_j.$$

Then we have

$$P^{(k)}_{[\varphi(\zeta)]}(\lambda) = v^{(k)}(\lambda) \cdot \varphi(\zeta).$$

Therefore

$$\frac{d}{d\zeta} P^{(k)}_{[\varphi(\zeta)]}(\lambda) = v^{(k)}(\lambda) \cdot \varphi'(\zeta).$$

So if $d^\lambda_{m_\lambda - k} \ge 2$, then

$$0 = \frac{d}{d\zeta}\bigg|_{\zeta=0} P^{(k)}_{[\varphi(\zeta)]}(\lambda) = v^{(k)}(\lambda) \cdot \varphi'(0). \tag{2.2}$$

Consider $d^\lambda_{m_\lambda - k} \ge 2$, this is equivalent to

$$F_0^\lambda \cap [m_\lambda - (m_\lambda - k) + 2..m_\lambda] \neq \emptyset.$$

This means that $b_s \ge k + 2$, so $k \in \{0, 1, \ldots, b_s - 2\}$.

On the other hand, $m_\lambda - (b_s - 1) = s_\lambda$ is the size of the biggest elementary Jordan block corresponding to λ appearing in B. We then obtain $n - m$ homogeneous linear equations of the type (2.2) for $\varphi'(0)$.

The final step of the first stage of the proof is to prove the linear independence of the coefficient vectors

$$\{v^{(k)}(\lambda) : \lambda \in \mathrm{Sp}(B), 0 \le k \le m_\lambda - s_\lambda - 1\}.$$

Denote by $[v_1, v_2, \ldots, v_n]$ the determinant of the matrix formed by $v_1, v_2, \ldots, v_n \in \mathbb{C}^n$. Then we find that

$$[v(\lambda_1), v(\lambda_2), \ldots, v(\lambda_n)]$$

is in fact Vandermonde's determinant, therefore

$$[v(\lambda_1), v(\lambda_2), \ldots, v(\lambda_n)] = \pm \prod_{i<j} (\lambda_j - \lambda_i).$$

So $v(\lambda_1), v(\lambda_2), \ldots, v(\lambda_n)$ are linearly independent if $\lambda_1, \lambda_2, \ldots, \lambda_n$ are distinct. To deal with derivatives $v^{(k)})(\lambda)$, we make use of divided differences and follow the notations in [7], precisely,

$$v[\lambda_1, \lambda_2] = \frac{v(\lambda_2) - v(\lambda_1)}{\lambda_2 - \lambda_1}.$$

We make first observations as follows:

$$[v(\lambda_1), v(\lambda_2), \ldots, v(\lambda_n)] = [v(\lambda_1), v(\lambda_2) - v(\lambda_1, v(\lambda_3), \ldots, v_{\lambda_n})]$$
$$= (\lambda_2 - \lambda_1)[v(\lambda_1), v[\lambda_1, \lambda_2], v(\lambda_3), \ldots, v(\lambda_n)].$$

Recall that $v'(\lambda_1) = \lim_{\lambda_2 \to \lambda_1} v[\lambda_1, \lambda_2]$. Therefore we get

$$[v(\lambda_1), v'(\lambda_1), v(\lambda_3), \ldots, v(\lambda_n)] = \pm \prod_{\substack{i<j \\ i,j \neq 3}} (\lambda_j - \lambda_i).$$

By induction and combining with Genocchi-Hermite formula [7] where

$$v^{(k)}(\lambda_1) = k! \lim_{\substack{\lambda_j \to \lambda_1 \\ j \neq 1}} v[\lambda_1, \lambda_2, \ldots, \lambda_{k+1}],$$

we deduce a formula of type

$$[v(\lambda_1), v'(\lambda_1), \ldots, v^{(k)}(\lambda_1), v(\lambda_2), \ldots, v(\lambda_{n-k})] = \pm 1! 2! \ldots k! \prod_{i<j}(\lambda_j - \lambda_i).$$

Therefore, we deduce that the coefficient vectors

$$\{v^{(k)}(\lambda) \ : \ \lambda \in \mathrm{Sp}(B), 0 \leq k \leq m_\lambda - s_\lambda - 1\}$$

are linearly independent, and we have such $n - m$ vectors. So the solution space is of dimension $\leq m$, it means

$$\mathrm{rank}(\pi'(B)) \leq m.$$

2.4.2. The rank is bigger than or equal to the degree

To realize this part of the proof, we construct m linearly independent vectors in the image space of $\pi'(B)$ in the echelon form as follows. The first ideas of the proof come from the last part of [8].

We suppose B is in the canonical form (or rational form), i.e.,

$$B = \begin{pmatrix} C_1 & & & \\ & C_2 & & \\ & & \ddots & \\ & & & C_l \end{pmatrix}$$

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where $C_i's$ are all companion matrices of the invariant factors of the characteristic polynomial of B and we arrange $C_i's$ in increasing order of size, and note that the largest size is of C_l and equal to m the degree of the minimal polynomial. On the other hand, C_l is the companion matrix of the minimal polynomial of B, i.e., C_l is of the form

$$C_l = \begin{pmatrix} 0 & 1 & & & \\ & 0 & 1 & & \\ & & \ddots & \ddots & \\ & & & 0 & 1 \\ -a_1 & a_2 & \cdots & -a_{m-1} & -a_m \end{pmatrix}.$$

Next put $H = \begin{pmatrix} H_1 & & & \\ & H_2 & & \\ & & \ddots & \\ & & & H_l \end{pmatrix}$ to be a diagonal block matrix of the same

form as B where $H_1, H_2, \ldots, H_{l-1}$ are zero matrices and H_l is of the form

$$H_l = \begin{pmatrix} 0 & 0 & & & 0 \\ & & \cdots & & \\ & & \cdots & & \\ -h_1 & -h_2 & \cdots & -h_{m-1} & -h_m \end{pmatrix},$$

i.e., only the last row of H_l has nonzero entries and $h_1, \ldots, h_m$ are complex variables.

Then $\sigma_k(B + \zeta H)$ is a linear polynomial of $h_m, h_{m-1}, \ldots, h_{m-k+1}$ (with nonzero coefficients on these variables) for $1 \leq k \leq m$.

Therefore if we let $h_i = 1$ and $h_j = 0$ for $j \neq i$ and $1 \leq j \leq m$, these vectors

$$\pi'(B)H = \frac{d}{d\zeta}\bigg|_{\zeta=0} \pi(B + \zeta H)$$

form m vectors in the echelon form, and thus, are linearly independent. It means

$$\operatorname{rank}(\pi'(B)) \geq m. \quad \square$$

3. Conclusions

We proved in this note that rank of $\pi'(B)$ is equal to the degree of the minimal polynomial of B. This result explains in part why we often have nice results if B is a non-derogatory matrix (or cyclic matrix according to the terminology of the authors in [2] or [4]) or in other words, cyclic matrices are regular points of π in the differential sense. This result partially explains why we can always lift the mapping $\varphi \colon \mathbb{D} \to \mathbb{G}_n$ locally in [3] (although these authors prove a stronger result: they prove the global lifting).

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