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# A VARIABLE METRIC INERTIAL FORWARD-REFLECTED-BACKWARD METHOD FOR SOLVING MONOTONE INCLUSIONS

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**Abstract.** We propose a new method for finding a zero point of a sum involving a Lipschitzian monotone operator and a maximally monotone operator, both acting on a real Hilbert space. The proposed method aims to extend forward-reflected-backward method by using inertial effect and variable metric. The weak convergence of the proposed method is proved under standard conditions.
***Keywords:*** monotone inclusion, forward-reflected-backward method, variable metric, inertial effect.

## 1. Introduction

Many important issues in operator theory, fixed point theorems, equilibrium problems, variational inequalities, convex optimization, image processing, or machine learning, reduce to the problem of solving monotone inclusions involving Lipschitzian operators (see [1-6] and the references therein). In this work, we consider the monotone inclusions of finding a zero point of sum of a maximal monotone operator $A$ and a monotone, $L$-Lipschitzian operator $B$, acting on a real Hilbert space $\mathcal{H}$, i.e.,

$$\text{Find } x \in \mathcal{H} \text{ such that } 0 \in (A+B)x. \tag{1.1}$$

Throughout this paper, we assume that a solution $x$ exists. For solving problem (1.1), several methods have been proposed. The first one is the forward-backward-forward method proposed by Tseng [7]:

$$\gamma \in \left]0, +\infty\right[, \quad \begin{cases} y_k = J_{\gamma A}(x_k - \gamma B x_k) \\ x_{k+1} = y_k - \gamma B y_k + \gamma B x_k. \end{cases}$$

A limitation of this method is that at each iteration step, one has to compute twice the values of operator $B$. This issue was recently resolved in [6], the forward reflected backward splitting method was proposed, namely,

$$\gamma \in \left]0, +\infty\right[, \quad x_{k+1} = J_{\gamma A}(x_k - 2\gamma B x_k + \gamma B x_{k-1}). \tag{1.2}$$

The convergence of (1.2) is derived under condition that $\gamma < \frac{1}{2L}$. In [5], for solving problem (1.1), an alternative method was proposed, namely the reflected-forward-backward method. We notice here that the methods in [5-7] are limited to the fixed metric. While the variable metric methods have obtained a lot of attention in the literature (see [8-12] and the references therein). Variable metric methods improve the convergence profiles.

In this paper, we consider a new splitting method for solving problem (1.1). The proposed method extends the forward-reflected-backward in [6] by using variable metric and inertial effect. In [13], Polyak introduced the so-called heavy ball method in order to speed up the classical gradient method. This idea was employed and refined by some authors. In [14], Alvarez and Attouch employed the heavy ball method and proposed the inertial proximal point algorithm. We emphasize that use of inertial effects helps increase the convergence rate of the algorithm [15, 16].

# 2. Preliminaries

## 2.1. Notations

We recall some notation and background from convex analysis and monotone operator theory (see [1] for detail).

The scalar products and the associated norms of a Hilbert space $\mathcal{H}$ are denoted respectively by $\langle \cdot \mid \cdot \rangle$ and $\|\cdot\|$. The symbols $\rightharpoonup$ and $\to$ denote respectively weak and strong convergence. We denote by $\mathcal{B}(\mathcal{H})$ the space of bounded linear operators from $\mathcal{H}$ in to itself and $\mathcal{S}(\mathcal{H}) = \{K \in \mathcal{B}(\mathcal{H}) | K = K^\star\}$, where $K^\star$ denotes the adjoint of $K$. The Loewner partial ordering on $\mathcal{S}(\mathcal{H})$ is defined as:

$$(\forall U \in \mathcal{S}(\mathcal{H}))(\forall V \in \mathcal{S}(\mathcal{H})) \;\; U \succcurlyeq V \iff (\forall x \in \mathcal{H}) \;\; \langle Ux \mid x \rangle \geq \langle Vx \mid x \rangle \, .$$

Let $\theta \geq 0$, we set

$$\mathcal{P}_\theta(\mathcal{H}) = \{U \in \mathcal{S}(\mathcal{H}) | \; U \succcurlyeq \theta \, \mathrm{Id}\},$$

for $U \in \mathcal{P}_\theta(\mathcal{H})$, we define a semi-scalar product and a semi-norm (a scalar product and a norm if $\theta > 0$) by:

$$(\forall (x, y) \in \mathcal{H}^2) \;\; \langle x \mid y \rangle_U = \langle Ux \mid y \rangle \text{ and } \|x\|_U = \sqrt{\langle Ux \mid x \rangle}.$$

Let $A \colon \mathcal{H} \to 2^{\mathcal{H}}$ be a set-valued operator. The domain of $A$ is denoted by $\mathrm{dom}(A)$ which is a set of all $x \in \mathcal{H}$ such that $Ax \neq \varnothing$. The range of $A$ is $\mathrm{ran}(A) =$

$\{u \in \mathcal{H} \mid (\exists\, x \in \mathcal{H}) u \in Ax\}$. The graph of $A$ is $\operatorname{gra}(A) = \{(x, u) \in \mathcal{H} \times \mathcal{H} \mid u \in Ax\}$. The inverse of $A$ is $A^{-1} \colon u \mapsto \{x \mid u \in Ax\}$. The zero set of $A$ is $\operatorname{zer}(A) = A^{-1}0$.

We denote as $\ell^1(\mathbb{N})$ the space of absolute summable sequences.

**Definition 2.1.** *We say that an operator $A \colon \mathcal{H} \to 2^{\mathcal{H}}$ is*

(i) *monotone if*

$$\big(\forall (x,u) \in \operatorname{gra}(A)\big)\big(\forall (y,v) \in \operatorname{gra}(A)\big) \quad \langle x - y \mid u - v\rangle \geq 0.$$

(ii) *maximally monotone if it is monotone and there exists no monotone operator $B \colon \mathcal{H} \to 2^{\mathcal{H}}$ such that $\operatorname{gra}(B)$ properly contains $\operatorname{gra}(A)$, i.e., there is no monotone operator that properly contains it.*

**Definition 2.2.** *A mapping $T : \mathcal{H} \to \mathcal{H}$ is said to be*

(i) *$L$-Lipschitz continuous ($L \in [0, +\infty[$) if*

$$\|Tx - Ty\| \leq L\|x - y\| \quad \forall x, y \in \mathcal{H}.$$

(ii) *$c$-cocoercive ($c \in [0, +\infty[$) if*

$$\langle x - y \mid Tx - Ty\rangle \geq c\|Tx - Ty\|^2 \quad \forall x, y \in \mathcal{H}.$$

**Definition 2.3.** *For $A \colon \mathcal{H} \to 2^{\mathcal{H}}$, the resolvent of operator $A$ is*

$$J_A = (\operatorname{Id} + A)^{-1},$$

*where $\operatorname{Id}$ denotes the identity operator on $\mathcal{H}$.*

Note that, when $A$ is maximally monotone, $J_A$ is an everywhere single-valued operator [1].

## 2.2. Technical results

The following properties can be found in [8, 9]:

**Lemma 2.1.** *[8, Lemma 2.1] Let $\theta \in ]0, +\infty[$, $\mu \in ]0, +\infty[$, and let $A, B \in \mathcal{S}(\mathcal{H})$ such that $\mu \operatorname{Id} \succcurlyeq A \succcurlyeq B \succcurlyeq \theta \operatorname{Id}$, then*

*(i)* $\theta^{-1} \operatorname{Id} \succcurlyeq B^{-1} \succcurlyeq A^{-1} \succcurlyeq \mu^{-1} \operatorname{Id}$.

*(ii)* $(\forall x \in \mathcal{H}) \;\; \langle A^{-1}x \mid x\rangle \geq \|A\|^{-1}\|x\|^2$.

*(iii)* $\|A^{-1}\| \leq \theta^{-1}$.

**Lemma 2.2.** *[8, Lemma 2.3] Let* $\theta \in ]0, +\infty[$*, let* $(\eta_k)_{k\in\mathbb{N}} \in \ell^1(\mathbb{N})$ *and let* $(W_k)_{k\in\mathbb{N}}$ *be a sequence in* $\mathcal{P}_\theta(\mathcal{H})$ *such that* $\mu = \sup_{k\in\mathbb{N}} \|W_k\| < +\infty$*. Assume that one of the following holds:*

*(i)* $(\forall k \in \mathbb{N})\ (1+\eta_k)W_k \succcurlyeq W_{k+1}$.

*(ii)* $(\forall k \in \mathbb{N})\ (1+\eta_k)W_{k+1} \succcurlyeq W_k$.

*Then there exists* $W \in \mathcal{P}_\theta(\mathcal{H})$ *such that* $W_k \to W$ *pointwise.*

**Lemma 2.3.** *[9, Lemma 3.7] Let* $A : \mathcal{H} \to 2^{\mathcal{H}}$ *be a maximally monotone,* $\theta \in ]0, +\infty[$*,* $U \in \mathcal{P}_\theta(\mathcal{H})$ *and* $\mathcal{G}$ *be the real Hilbert space obtained by endowing* $\mathcal{H}$ *with the scalar product* $(x, y) \mapsto \langle x \mid y\rangle_{U^{-1}} = \langle x \mid U^{-1}y\rangle$*. The following properties hold:*

*(i)* $UA : \mathcal{G} \to 2^{\mathcal{G}}$ *is maximally monotone.*

*(ii)* $J_{UA} : \mathcal{G} \to 2^{\mathcal{G}}$ *is* $1-$*cocoercive.*

We also have the following results which are necessary to prove the convergence of the algorithm:

**Lemma 2.4.** *Assume* $(z_k)_{k\in\mathbb{N}}$*,* $(\alpha_k)_{k\in\mathbb{N}}$*,* $(t_k)_{k\in\mathbb{N}}$ *are nonnegative sequences such that* $(\alpha_k)_{k\in\mathbb{N}}$ *is summable and*

$$(\forall k \in \mathbb{N})\ \ z_{k+1} \le (1+\alpha_k)z_k - t_k.$$

*then* $(z_k)_{k\in\mathbb{N}}$ *converges and* $(t_k)_{k\in\mathbb{N}}$ *is summable.*

**Lemma 2.5.** *Let* $(\alpha_k)_{k\in\mathbb{N}}$*,* $(\theta_k)_{k\in\mathbb{N}}$*,* $(\lambda_k)_{k\in\mathbb{N}}$ *be sequences in* $\mathbb{R}$*. Assume that* $(\lambda_k)_{k\in\mathbb{N}}$ *is absolutely summable sequence and there exists* $t > 0$ *such that* $\alpha_k \ge (1+t)|\theta_k|$*,* $\forall k \in \mathbb{N}$*. Then, there exist a* $t_0 > 0$ *and* $k_0 \in \mathbb{N}$ *such that* $\forall k \ge k_0$*, we have*

$$\frac{\alpha_k}{1+\lambda_k} + \theta_k \ge \frac{\alpha_k + \theta_k}{1+\lambda_k(1+\frac{1}{t_0})}. \tag{2.1}$$

*Proof.* (2.1) is equivalent to

$$\begin{aligned}
&\big(1+\lambda_k(1+\frac{1}{t_0})\big)(\alpha_k + (1+\lambda_k)\theta_k) \ge (1+\lambda_k)(\alpha_k+\theta_k)\\
&\Leftrightarrow \frac{\lambda_k}{t_0}\alpha_k + \lambda_k(1+\lambda_k)(1+\frac{1}{t_0})\theta_k \ge 0\\
&\Leftrightarrow \alpha_k + (1+\lambda_k)(1+t_0)\theta_k \ge 0.
\end{aligned}$$

There exists a $k_0 \in \mathbb{N}$ such that $\forall k \ge k_0$: $|\lambda_k| \le \frac{t}{2}$, then for $t_0 \le \frac{t}{t+2}$, we have

$$|(1+\lambda_k)(1+t_0)| \le (1+\frac{t}{2})(1+\frac{t}{t+2}) = t+1.$$

Hence, we obtain the desired result from the condition that $\alpha_k \ge (1+t)|\theta_k|$.

□

# 3. Proposed method and convergences

## 3.1. Proposed method and related works

We develop the following variable metric framework for solving problem (1.1). Let $\gamma > 0$, $\alpha \geq 0$, $(\eta_k)_{k\in\mathbb{N}}$ be a non-negative sequence in $\ell^1(\mathbb{N})$ and $(U_k)_{k\in\mathbb{N}}$ be a sequence in $\mathcal{P}_\theta(\mathcal{H})$ ($\theta > 0$). Assume that

$$\mu = \sup_{k\in\mathbb{N}} \|U_k\| < +\infty \quad \text{and } (\forall k \in \mathbb{N})\ (1+\eta_k)U_{k+1} \succcurlyeq U_k \succcurlyeq U_{k+1}. \tag{3.1}$$

Let $x_{-1},\ x_0 \in \mathcal{H}$, we set $(\forall k \in \mathbb{N})$

$$x_{k+1} = J_{\gamma U_k A}\big[(1+\alpha)x_k - \alpha x_{k-1} - \gamma(2U_kBx_k - U_kBx_{k-1})\big]. \tag{3.2}$$

**Remark 3.1.** (i) *The condition* (3.1) *was introduced in [8] and utilized in [9, 11, 12]. We also note that condition is satisfied in particular when $U_k \equiv U \in \mathcal{P}_\theta(\mathcal{H})$ and $\eta_k = 0$ $(\forall k \in \mathbb{N})$. In case variable metric is not constant, we can choose $U_{k+1} = \frac{U_k}{1+\eta_k}$.*

(ii) *In the case when* $(\forall k \in \mathbb{N})$ $U_k = \mathrm{Id}$, $\alpha = 0$, *then* (3.2) *becomes the forward-reflected-backward method proposed in [6].*

**Remark 3.2.** *Variable metric algorithms have a long history. Variable metric methods in optimization were introduced in [10, 17] to improve the convergence profiles. The idea was then extended to the variable metric proximal point algorithm to find a zero point of a maximal monotone operator; see [8, 18, 19] for instances. For the problem of finding a zero point of the sum of a maximally monotone operator and a Lipschitzian monotone operator, the variable metric methods were developed in [11, 12].*

## 3.2. Weak convergence

To prove the convergence of Algorithm 3.1., we need the following lemma.

**Lemma 3.1.** *Let $(x_k)_{k\in\mathbb{N}}$ be a sequence generated from Algorithm 3.1., and $x \in \mathrm{zer}(A + B)$, then we have*

$$\begin{aligned}
&\|x_k - x\|^2_{U_k^{-1}} + (\gamma\mu L + 2\alpha)\|x_k - x_{k-1}\|^2_{U_k^{-1}} - \alpha\|x_{k-1} - x\|^2_{U_k^{-1}} \\
&+ 2\gamma\,\langle Bx_{k-1} - Bx_k \mid x_k - x\rangle \\
&\geq \frac{1}{1+\eta_k}\left(\|x_{k+1} - x\|^2_{U_{k+1}^{-1}} + (\gamma\mu L + 2\alpha)\|x_{k+1} - x_k\|^2_{U_{k+1}^{-1}}\right) - \alpha\|x_k - x\|^2_{U_{k+1}^{-1}} \\
&+ 2\gamma\,\langle Bx_k - Bx_{k+1} \mid x_{k+1} - x\rangle + (1 - 3\alpha - 2\gamma\mu L)\|x_{k+1} - x_k\|^2_{U_k^{-1}}.
\end{aligned} \tag{3.3}$$

*Proof.* From (3.2), we get

$$(1+\alpha)x_k - \alpha x_{k-1} - \gamma(2U_kBx_k - U_kBx_{k-1}) - x_{k+1} \in \gamma U_k A x_{k+1},$$

then we have

$$\begin{aligned}&\frac{U_k^{-1}\big((1+\alpha)x_k-\alpha x_{k-1}-x_{k+1}\big)}{\gamma}-2Bx_k+Bx_{k-1}\in Ax_{k+1},\\&-Bx\in Ax,\end{aligned}$$

hence

$$\left\langle\frac{U_k^{-1}\big((1+\alpha)x_k-\alpha x_{k-1}-x_{k+1}\big)}{\gamma}-2Bx_k+Bx_{k-1}+Bx\mid x_{k+1}-x\right\rangle\geq 0,$$

which implies

$$\begin{aligned}&\left\langle\frac{U_k^{-1}(x_k-x_{k+1})}{\gamma}\mid x_{k+1}-x\right\rangle+\alpha\left\langle\frac{U_k^{-1}(x_k-x_{k-1})}{\gamma}\mid x_{k+1}-x_k+x_k-x\right\rangle\\&\geq\langle 2Bx_k-Bx_{k-1}\mid x_{k+1}-x\rangle,\end{aligned}$$

which is equivalent to

$$\begin{aligned}&\frac{1}{2\gamma}\big(\|x_k-x\|^2_{U_k^{-1}}-\|x_{k+1}-x\|^2_{U_k^{-1}}-\|x_{k+1}-x_k\|^2_{U_k^{-1}}\big)\\&-\frac{\alpha}{2\gamma}\big(\|x_{k-1}-x\|^2_{U_k^{-1}}-\|x_k-x\|^2_{U_k^{-1}}-\|x_{k-1}-x_k\|^2_{U_k^{-1}}\big)\\&+\frac{\alpha}{\gamma}\left\langle U_k^{-1}(x_k-x_{k-1})\mid x_{k+1}-x_k\right\rangle\\&\geq\langle Bx_k-Bx_{k+1}\mid x_{k+1}-x\rangle-\langle Bx_{k-1}-Bx_k\mid x_{k+1}-x_k\rangle\\&-\langle Bx_{k-1}-Bx_k\mid x_k-x\rangle.\end{aligned}\tag{3.4}$$

By utilizing the Cauchy-Schwarz inequality and Lemma 2.1 (i), we have

$$|2\left\langle U_k^{-1}(x_k-x_{k-1})\mid x_{k+1}-x_k\right\rangle|\leq\|x_k-x_{k-1}\|^2_{U_k^{-1}}+\|x_{k+1}-x_k\|^2_{U_k^{-1}},$$

and

$$\begin{aligned}|2\langle Bx_{k-1}-Bx_k\mid x_{k+1}-x_k\rangle|&\leq L\big(\|x_{k-1}-x_k\|^2+\|x_{k+1}-x_k\|^2\big)\\&\leq\mu L(\|x_k-x_{k-1}\|^2_{U_k^{-1}}+\|x_{k+1}-x_k\|^2_{U_k^{-1}}).\end{aligned}$$

Inequality (3.4) implies that

$$\begin{aligned}&\big(\|x_k-x\|^2_{U_k^{-1}}-\|x_{k+1}-x\|^2_{U_k^{-1}}-\|x_{k+1}-x_k\|^2_{U_k^{-1}}\big)\\&-\alpha\big(\|x_{k-1}-x\|^2_{U_k^{-1}}-\|x_k-x\|^2_{U_k^{-1}}-\|x_{k-1}-x_k\|^2_{U_k^{-1}}\big)\\&+\alpha(\|x_k-x_{k-1}\|^2_{U_k^{-1}}+\|x_{k+1}-x_k\|^2_{U_k^{-1}})\\&\geq 2\gamma\langle Bx_k-Bx_{k+1}\mid x_{k+1}-x\rangle-2\gamma\langle Bx_{k-1}-Bx_k\mid x_k-x\rangle\\&-\gamma\mu L(\|x_k-x_{k-1}\|^2_{U_k^{-1}}+\|x_{k+1}-x_k\|^2_{U_k^{-1}}).\end{aligned}$$

We obtain

$$
\begin{aligned}
&\|x_k - x\|^2_{U_k^{-1}} - \alpha\|x_{k-1} - x\|^2_{U_k^{-1}} + (\gamma\mu L + 2\alpha)\|x_k - x_{k-1}\|^2_{U_k^{-1}} \\
&+ 2\gamma \langle Bx_{k-1} - Bx_k \mid x_k - x\rangle \\
&\geq \|x_{k+1} - x\|^2_{U_k^{-1}} - \alpha\|x_k - x\|^2_{U_k^{-1}} + (\gamma\mu L + 2\alpha)\|x_{k+1} - x_k\|^2_{U_k^{-1}} \\
&+ 2\gamma \langle Bx_k - Bx_{k+1} \mid x_{k+1} - x\rangle + (1 - 3\alpha - 2\gamma\mu L)\|x_{k+1} - x_k\|^2_{U_k^{-1}}.
\end{aligned}
$$

From (3.1) and Lemma 2.1 (i), we have $(1+\eta_k)U_k^{-1} \succcurlyeq U_{k+1}^{-1} \succcurlyeq U_k^{-1}$, we obtain (3.3). The proof is completed. □

We have the following theorem.

**Theorem 3.1.** *Let $(x_k)_{k\in\mathbb{N}}$ be a sequence generated from Algorithm 3.1. with $\alpha \in [0, \frac{1}{3}[$. Suppose that*

$$
\gamma < \frac{1 - 3\alpha}{2\mu L}. \tag{3.5}
$$

*Then $(x_k)_{k\in\mathbb{N}}$ converges weakly to a point in* $\operatorname{zer}(A + B)$.

*Proof.* We rewrite (3.3) as

$$
\begin{aligned}
&\|x_k - x\|^2_{U_k^{-1}} + (\gamma\mu L + 2\alpha)\|x_k - x_{k-1}\|^2_{U_k^{-1}} - \alpha\|x_{k-1} - x\|^2_{U_k^{-1}} \\
&+ 2\gamma \langle Bx_{k-1} - Bx_k \mid x_k - x\rangle \\
&\geq \frac{1}{1+\eta_k}\left(\|x_{k+1} - x\|^2_{U_{k+1}^{-1}} + (\gamma\mu L + 2\alpha)\|x_{k+1} - x_k\|^2_{U_{k+1}^{-1}}\right) - \alpha\|x_k - x\|^2_{U_{k+1}^{-1}} \\
&+ 2\gamma \langle Bx_k - Bx_{k+1} \mid x_{k+1} - x\rangle + (1 - 3\alpha - 2\gamma\mu L)\|x_{k+1} - x_k\|^2_{U_k^{-1}}.
\end{aligned}
$$

We set

$$
T_k = \|x_k - x\|^2_{U_k^{-1}} + (\gamma\mu L + 2\alpha)\|x_k - x_{k-1}\|^2_{U_k^{-1}}. \tag{3.6}
$$

Then we have

$$
\begin{aligned}
&T_k - \alpha\|x_{k-1} - x\|^2_{U_k^{-1}} + 2\gamma \langle Bx_{k-1} - Bx_k \mid x_k - x\rangle \\
&\geq \frac{1}{1+\eta_k}T_{k+1} - \alpha\|x_k - x\|^2_{U_{k+1}^{-1}} + 2\gamma \langle Bx_k - Bx_{k+1} \mid x_{k+1} - x\rangle \\
&+ (1 - 3\alpha - 2\gamma\mu L)\|x_{k+1} - x_k\|^2_{U_k^{-1}}.
\end{aligned} \tag{3.7}
$$

We will show that there exists a $t > 0$ such that

$$
T_k \geq (1+t)| - \alpha\|x_{k-1} - x\|^2_{U_k^{-1}} + 2\gamma \langle Bx_{k-1} - Bx_k \mid x_k - x\rangle |.
$$

Indeed, from the definition of $T_k$ (3.6), we have

$$\begin{aligned}T_k &= (1-3\alpha-2\gamma\mu L)\|x_k - x\|^2_{U_k^{-1}} + (3\alpha+2\gamma\mu L)\|x_k - x\|^2_{U_k^{-1}} \\ &\quad + (\gamma\mu L + 2\alpha)\|x_k - x_{k-1}\|^2_{U_k^{-1}} \\ &= (1-3\alpha-2\gamma\mu L)\|x_k - x\|^2_{U_k^{-1}} + \alpha(3\|x_k - x\|^2_{U_k^{-1}} + 2\|x_k - x_{k-1}\|^2_{U_k^{-1}}) \\ &\quad + \gamma\mu L(2\|x_k - x\|^2_{U_k^{-1}} + \|x_k - x_{k-1}\|^2_{U_k^{-1}})\end{aligned}$$

Using $U_k^{-1} \succcurlyeq \frac{1}{\mu}\,\mathrm{Id}$, the inequality $3x^2 + y^2 \geq \frac{6}{5}(x+y)^2$, the Lipschitz condition of $B$ and the Cauchy-Schwarz inequality, we get

$$\begin{aligned}T_k &\geq (1-3\alpha-2\gamma\mu L)\|x_k - x\|^2_{U_k^{-1}} + \frac{6\alpha}{5}\|x_{k-1} - x\|^2_{U_k^{-1}} \\ &\quad + \gamma L(2\|x_k - x\|^2 + \|x_k - x_{k-1}\|^2) \\ &\geq (1-3\alpha-2\gamma\mu L)\|x_k - x\|^2_{U_k^{-1}} + \frac{6\alpha}{5}\|x_{k-1} - x\|^2_{U_k^{-1}} \\ &\quad + 2\gamma L\sqrt{2}|\left\langle x_k - x \mid x_k - x_{k-1}\right\rangle| \\ &\geq (1-3\alpha-2\gamma\mu L)\|x_k - x\|^2_{U_k^{-1}} + \frac{6}{5}| - \alpha\|x_{k-1} - x\|^2_{U_k^{-1}} \\ &\quad + 2\gamma\left\langle Bx_{k-1} - Bx_k \mid x_k - x\right\rangle|. \end{aligned} \tag{3.8}$$

Hence, Lemma 2.5 and (3.7) imply that there exist $t_0 > 0$ and $k_0 \in \mathbb{N}$ such that $\forall k \geq k_0$. By this, we obtain

$$S_k \geq \frac{1}{1 + (1+\frac{1}{t_0})\eta_k} S_{k+1} + (1-3\alpha-2\gamma\mu L)\|x_{k+1} - x_k\|^2_{U_k^{-1}}, \tag{3.9}$$

where $S_k = T_k - \alpha\|x_{k-1} - x\|^2_{U_k^{-1}} + 2\gamma\left\langle Bx_{k-1} - Bx_k \mid x_k - x\right\rangle$.
From condition (3.5) and (3.8), we deduce

$$S_k \geq (1-3\alpha-2\gamma\mu L)\|x_k - x\|^2_{U_k^{-1}} \geq 0. \tag{3.10}$$

It follows from (3.9) that

$$S_{k+1} \leq (1 + (1+\frac{1}{t_0})\eta_k)S_k - (1-3\alpha-2\gamma\mu L)\|x_{k+1} - x_k\|^2_{U_k^{-1}}.$$

Note that $(\eta_k)_{k\in\mathbb{N}} \in \ell^1(\mathbb{N})$, then using Lemma 2.4, we obtain

$$\begin{cases}\|x_{k+1} - x_k\| \to 0, \\ \text{there exists } \xi \in \mathbb{R} \text{ such that } S_k \to \xi.\end{cases}$$

From $S_k \to \xi$ and (3.10) it can be deduced that the sequence $(x_k)_{k\in\mathbb{N}}$ is bounded. Therefore

$$\begin{aligned}
&\lim_{k\to\infty} S_k \\
&= \lim_{k\to\infty} \Big( \|x_k - x\|^2_{U_k^{-1}} + (\gamma\mu L + 2\alpha)\|x_k - x_{k-1}\|^2_{U_k^{-1}} - \alpha\|x_{k-1} - x\|^2_{U_k^{-1}} \\
&\qquad + 2\gamma \langle Bx_{k-1} - Bx_k \mid x_k - x\rangle \Big) \\
&= \lim_{k\to\infty} \big(\|x_k - x\|^2_{U_k^{-1}} + -\alpha\|x_{k-1} - x\|^2_{U_k^{-1}}\big) \\
&= \xi.
\end{aligned}$$

Let $(x^\star, u^\star)$ be a weak sequential cluster point of $(x_k, u_k)_{k\in\mathbb{N}}$. Then there exists a subsequence $(x_{k_n}, u_{k_n})_{n\in\mathbb{N}}$ that converges weakly to $(x^\star, u^\star)$.
We have

$$\frac{U_k^{-1}\big((1+\alpha)x_k - \alpha x_{k-1} - x_{k+1}\big)}{\gamma} - 2Bx_k + Bx_{k-1} \in Ax_{k+1},$$

which implies

$$\frac{U_k^{-1}\big((1+\alpha)x_k - \alpha x_{k-1} - x_{k+1}\big)}{\gamma} - 2Bx_k + Bx_{k-1} + Bx_{k+1} \in (A+B)x_{k+1}. \tag{3.11}$$

We have

$$\begin{aligned}
\|(1+\alpha)x_k - \alpha x_{k-1} - x_{k+1}\| &= \|(x_k - x_{k+1}) + \alpha(x_k - x_{k-1})\| \\
&\le \|x_{k+1} - x_k\| + \alpha\|x_k - x_{k-1}\|,
\end{aligned}$$

and

$$\begin{aligned}
\|-2Bx_k + Bx_{k-1} + Bx_{k+1}\| &\le \|Bx_{k+1} - Bx_k\| + \|Bx_k - Bx_{k-1}\| \\
&\le L(\|x_{k+1} - x_k\| + \|x_k - x_{k-1}\|).
\end{aligned}$$

We deduce that the left-hand side of (3.11) converges strongly to $0$. Thus, the right-hand side of (3.11) is a maximal monotone operator on $\mathcal{H}$ and $\mathrm{gra}(A+B)$ is closed under $\mathcal{H}^{\text{weak}} \times \mathcal{H}^{\text{strong}}$ [1]. Therefore, $0 \in (A+B)x^\star$ which implies $x^\star \in \mathrm{zer}(A+B)$.

Assume that $(x_{k_n})_{n\in\mathbb{N}} \rightharpoonup x$, $(x_{l_n})_{n\in\mathbb{N}} \rightharpoonup y$. We have that

$$\begin{aligned}
&-\big(\|x_k - x\|^2_{U_k^{-1}} - \alpha\|x_{k-1} - x\|^2_{U_k^{-1}}\big) + \big(\|x_k - y\|^2_{U_k^{-1}} - \alpha\|x_{k-1} - y\|^2_{U_k^{-1}}\big) \\
&+ (\|x\|^2_{U_k^{-1}} - \alpha\|x\|^2_{U_k^{-1}}) - (\|y\|^2_{U_k^{-1}} - \alpha\|y\|^2_{U_k^{-1}}) \\
&= 2\big(\langle x_k \mid x - y\rangle_{U_k^{-1}} - \alpha\langle x_{k-1} \mid x - y\rangle_{U_k^{-1}}\big).
\end{aligned} \tag{3.12}$$

Choose $k = k_n$ and $k = l_n$ then take limit both sides of (3.12) when $n \to \infty$, note that, by using Lemma 2.2, we have

$$U_k \to U \in \mathcal{P}_\theta.$$

We obtain

$$\|x - y\|^2_{U^{-1}} - \alpha\|x - y\|^2_{U^{-1}} = 0,$$

which implies that $x = y$. Hence $(x_k)_{k\in\mathbb{N}}$ converges weakly to $x$. The proof is completed. □

**Remark 3.3.** *When* $\alpha = 0,\ U_k = \mathrm{Id}\ (\forall k \in \mathbb{N})$*, the condition* (3.5) *of stepsize becomes* $\gamma < \frac{1}{2L}$*. This result recovers the result in [6].*

# 4. Conclusions

The paper proposed a novel variable metric inertial method for locating the zero point of a sum involving two operators. Specifically, one operator is maximally monotone, while the other is monotone-Lipschitz. By satisfying certain conditions on the parameters and metrics, we rigorously established the weak convergence of the proposed algorithm.

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