HNUE JOURNAL OF SCIENCE
Natural Science, 2024, Volume 69, Issue 3, pp. 3-13
This paper is available online at http://hnuejs.edu.vn/ns
DOI: 10.18173/2354-1059.2024-0030

A NEW SPLITTING METHOD FOR MONOTONE INCLUSIONS

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Received September 17, 2024. Revised October 23, 2024. Accepted October 30, 2024.

Abstract. In this paper, we propose a splitting method for finding a zero point of the sum of two operators in Hilbert spaces. Our method is a modification of the forward-backward algorithm by using the inertial effect. Under the imposed condition for parameters, weak convergence of the iterative sequence is established. We also give some numerical experiments to demonstrate the efficiency of the proposed algorithm.
Keywords: monotone inclusion, splitting method, inertial effect, forward-backward algorithm.

1. Introduction

In this paper, we consider the problem of finding zero points of the sum of a maximal monotone operator A and a monotone, $L-$Lipschitzian operator B, acting on a real Hilbert space $\mathcal{H}$. The problem is specified as

$$\text{find } \overline{x} \in \mathcal{H} \text{ such that } 0 \in (A+B)\overline{x}. \tag{1.1}$$

Throughout this paper, we assume that a solution $\overline{x}$ exists. This inclusion arises in numerous problems in monotone operator theory, variational inequalities, convex optimization, equilibrium problems, image processing, and machine learning; see [1]-[10] and the references therein.

There are many methods for solving problem (1.1). These methods exploit the splitting structure of (1.1) to use individual operators A and B. Classical methods include gradient, extragradient, past-extragradient, proximal-point, forward-backward splitting, forward-backward-forward splitting, Douglas-Rachford splitting, forward-reflected-backward splitting, reflected-forward-backward splitting, golden ratio, projective splitting methods, and their variants, see for examples [5], [11]-[15] for more details.

In this paper, we design a new method for solving problem (1.1). Our idea is based on the forward-reflected-backward method, which is presented by Malitsky and Tam in [14] and Polyak's inertial technique [16]. In [14], the authors proposed the forward-reflected-backward method (FRB), namely,

$$\gamma \in (0, +\infty), \quad x_{k+1} = J_{\gamma A}(x_k - 2\gamma B x_k + \gamma B x_{k-1}). \tag{1.2}$$

where J_A denotes the resolvent of A, i.e.

$$J_A = (\mathrm{Id} + A)^{-1},$$

and Id is the identity operator on $\mathcal{H}$.

In [16], Polyak introduced the so-called heavy ball method in order to speed up the classical gradient method. For a differential function $f\colon \mathcal{H} \to \mathbb{R}$, the algorithm takes the following form:

$$x_{k+1} = x_k + \alpha_k(x_k - x_{k-1}) - \gamma_k \nabla f(x_k).$$

This idea is then employed and refined by some authors [17]-[20]. Our method also uses the inertial effect to improve the performance of the algorithm. Unlike (1.2) which use two values of operator B at each iteration, we use three values of B in each iteration. Under some standard conditions, we also obtain the convergence of the proposed method. In some examples, our method gives better convergence rate than Tseng's method [12] and the FRB method in [14].

2. Preliminaries

The scalar product and the associated norm of the real Hilbert space $\mathcal{H}$ are denoted respectively by $\langle \cdot \mid \cdot \rangle$ and $\|\cdot\|$.

The symbols $\rightharpoonup$ and $\to$ denote respectively weak and strong convergence.

Let $A\colon \mathcal{H} \to 2^{\mathcal{H}}$ be a set-valued operator. The domain of A, denoted by $\mathrm{dom}(A)$, is set of all $x \in \mathcal{H}$ such that $Ax \neq \varnothing$. The range of A is defined by $\mathrm{ran}(A) = \big\{u \in \mathcal{H} \mid (\exists\, x \in \mathcal{H}) u \in Ax\big\}$. The graph of A is denoted by $\mathrm{gra}(A) = \big\{(x,u) \in \mathcal{H} \times \mathcal{H} \mid u \in Ax\big\}$. A^{-1} stands for the inverse of A, i.e., $A^{-1}\colon u \mapsto \big\{x \mid u \in Ax\big\}$. The zero set of A is $\mathrm{zer}(A) = A^{-1}0$.

Definition 2.1. *[11] We say that operator $A\colon \mathcal{H} \to 2^{\mathcal{H}}$ is*

1. *monotone if*

$$\big(\forall (x,u) \in \mathrm{gra}(A)\big)\big(\forall (y,v) \in \mathrm{gra}(A)\big) \quad \langle x - y \mid u - v\rangle \geq 0.$$

2. *maximally monotone if it is monotone and there exists no monotone operator $B\colon \mathcal{H} \to 2^{\mathcal{H}}$ such that* $\operatorname{gra}(B)$ *properly contains* $\operatorname{gra}(A)$, *i.e., there is no monotone operator that properly contains it.*

Definition 2.2. *[11] A mapping $T : \mathcal{H} \to \mathcal{H}$ is said to be L-Lipschitz continuous $(L > 0)$ if*

$$\|Tx - Ty\| \leq L\|x - y\| \quad \forall x, y \in \mathcal{H}.$$

Definition 2.3. *[11] For $A\colon \mathcal{H} \to 2^{\mathcal{H}}$, the resolvent of operator A is*

$$J_A = (\operatorname{Id} + A)^{-1},$$

where Id *denotes the identity operator on $\mathcal{H}$.*

Note that, when A is maximally monotone, J_A is an everywhere single-valued operator [11].

3. Proposed method and convergence

We propose the following method for solving problem (1.1).

Algorithm 3.1. *Let $\gamma > 0$, and $\alpha \geq 0$. Let $x_{-1},\ x_0,\ x_1 \in \mathcal{H}$. Iterate* $(\forall k \in \mathbb{N})$

$$x_{k+1} = J_{\gamma A}\Big[x_k + \alpha(x_k - x_{k-1}) - \gamma\Big(\frac{5}{2}Bx_k - 2Bx_{k-1} + \frac{1}{2}Bx_{k-2}\Big)\Big]. \tag{3.1}$$

Let us give some comments on the above algorithm.

1. In (3.1), in each iteration, we have to calculate three forward values. However, in the next iteration, we can use two forward values of the previous iteration. Therefore, we actually only compute one forward value in every iteration.

2. When $\alpha = 0,\ B = 0$, Algorithm 3.1 becomes the proximal point algorithm as in [21].

To prove the convergence of Algorithm 3.1, we need the following lemma.

Lemma 3.1. *Suppose that $(x_k)_{k\in\mathbb{N}}$ is the sequence generated by Algorithm 3.1. Then, for any $x \in \operatorname{zer}(A + B)$, we get*

$$\begin{aligned}&\|x_{k+1} - x\|^2 - \alpha\|x_k - x\|^2 + 2\gamma t_{k+1} + \big(1 - \alpha - \frac{3\gamma L}{2}\big)\|x_{k+1} - x_k\|^2\\ &\leq \|x_k - x\|^2 - \alpha\|x_{k-1} - x\|^2 + 2\gamma t_k + (2\alpha + \gamma L)\|x_k - x_{k-1}\|^2 + \frac{\gamma L}{2}\|x_{k-1} - x_{k-2}\|^2,\end{aligned} \tag{3.2}$$

where $t_k = \langle Bx_{k-1} - Bx_k \mid x_k - x\rangle + \frac{1}{2}\langle Bx_{k-1} - Bx_{k-2} \mid x_k - x\rangle$.

Proof. From (3.1), we get

$$\frac{x_k + \alpha(x_k - x_{k-1}) - x_{k+1}}{\gamma} - \Big(\frac{5}{2}Bx_k - 2Bx_{k-1} + \frac{1}{2}Bx_{k-2}\Big) \in Ax_{k+1}. \tag{3.3}$$

For $x \in \operatorname{zer}(A+B)$, then $-Bx \in Ax$. Hence,

$$\left\langle \frac{x_k + \alpha(x_k - x_{k-1}) - x_{k+1}}{\gamma} - \frac{5}{2}Bx_k + 2Bx_{k-1} - \frac{1}{2}Bx_{k-2} + Bx \mid x_{k+1} - x \right\rangle \geq 0,$$

which implies

$$\begin{aligned} &\left\langle \frac{x_k - x_{k+1} + \alpha(x_k - x_{k-1})}{\gamma} \mid x_{k+1} - x \right\rangle \\ &\geq \left\langle \frac{5}{2}Bx_k - 2Bx_{k-1} + \frac{1}{2}Bx_{k-2} - Bx \mid x_{k+1} - x \right\rangle. \end{aligned} \tag{3.4}$$

For the left-hand side of (3.4), we have

$$\begin{aligned} &\langle x_k - x_{k+1} + \alpha(x_k - x_{k-1}) \mid x_{k+1} - x\rangle \\ &= \langle x_k - x_{k+1} \mid x_{k+1} - x\rangle + \alpha \langle x_k - x_{k-1} \mid x_{k+1} - x_k\rangle + \alpha \langle x_k - x_{k-1} \mid x_k - x\rangle \\ &= \frac{1}{2}\big(\|x_k - x\|^2 - \|x_{k+1} - x\|^2 - \|x_{k+1} - x_k\|^2\big) \\ &\quad - \frac{\alpha}{2}\big(\|x_{k-1} - x\|^2 - \|x_k - x\|^2 - \|x_{k-1} - x_k\|^2\big) + \alpha \langle x_k - x_{k-1} \mid x_{k+1} - x_k\rangle. \end{aligned} \tag{3.5}$$

Using the monotonicity of B, we estimate the right-hand side of (3.4) as

$$\begin{aligned} &\left\langle \frac{5}{2}Bx_k - 2Bx_{k-1} + \frac{1}{2}Bx_{k-2} - Bx \mid x_{k+1} - x \right\rangle \\ &= \left\langle \frac{5}{2}Bx_k - 2Bx_{k-1} + \frac{1}{2}Bx_{k-2} - Bx - Bx_{k+1} \mid x_{k+1} - x \right\rangle \\ &\quad + \langle Bx_{k+1} - Bx \mid x_{k+1} - x\rangle \\ &\geq \langle Bx_k - Bx_{k+1} \mid x_{k+1} - x\rangle + \frac{3}{2}\langle Bx_k - Bx_{k-1} \mid x_{k+1} - x\rangle \\ &\quad - \frac{1}{2}\langle Bx_{k-1} - Bx_{k-2} \mid x_{k+1} - x\rangle \\ &= \langle Bx_k - Bx_{k+1} \mid x_{k+1} - x\rangle + \frac{1}{2}\langle Bx_k - Bx_{k-1} \mid x_{k+1} - x\rangle \\ &\quad + \langle Bx_k - Bx_{k-1} \mid x_{k+1} - x_k\rangle + \langle Bx_k - Bx_{k-1} \mid x_k - x\rangle \\ &\quad - \frac{1}{2}\Big(\langle Bx_{k-1} - Bx_{k-2} \mid x_{k+1} - x_k\rangle + \langle Bx_{k-1} - Bx_{k-2} \mid x_k - x\rangle \Big) \\ &= t_{k+1} - t_k + \langle Bx_k - Bx_{k-1} \mid x_{k+1} - x_k\rangle - \frac{1}{2}\langle Bx_{k-1} - Bx_{k-2} \mid x_{k+1} - x_k\rangle. \end{aligned} \tag{3.6}$$

Hence, from (3.4), (3.5) and (3.6), we can deduce

$$\begin{aligned}
&\Big(\|x_k - x\|^2 - \|x_{k+1} - x\|^2 - \|x_{k+1} - x_k\|^2\Big)\\
&\quad - \alpha\Big(\|x_{k-1} - x\|^2 - \|x_k - x\|^2 - \|x_{k-1} - x_k\|^2\Big)\\
&\geq 2\gamma t_{k+1} - 2\gamma t_k + 2\gamma \langle Bx_k - Bx_{k-1} \mid x_{k+1} - x_k\rangle\\
&\quad - \gamma \langle Bx_{k-1} - Bx_{k-2} \mid x_{k+1} - x_k\rangle - 2\alpha \langle x_k - x_{k-1} \mid x_{k+1} - x_k\rangle . \qquad (3.7)
\end{aligned}$$

Using Cauchy-Schwarz inequality and the Lipschitz property of B, we have

$$\begin{cases}
2|\langle Bx_k - Bx_{k-1} \mid x_{k+1} - x_k\rangle| \leq L(\|x_k - x_{k-1}\|^2 + \|x_{k+1} - x_k\|^2),\\
2|\langle Bx_{k-1} - Bx_{k-2} \mid x_{k+1} - x_k\rangle| \leq L(\|x_{k-1} - x_{k-2}\|^2 + \|x_{k+1} - x_k\|^2),\\
2|\langle x_k - x_{k-1} \mid x_{k+1} - x_k\rangle| \leq \|x_k - x_{k-1}\|^2 + \|x_{k+1} - x_k\|^2.
\end{cases}$$

Therefore, (3.7) implies that

$$\begin{aligned}
&\Big(\|x_k - x\|^2 - \|x_{k+1} - x\|^2 - \|x_{k+1} - x_k\|^2\Big)\\
&\quad - \alpha\Big(\|x_{k-1} - x\|^2 - \|x_k - x\|^2 - \|x_{k-1} - x_k\|^2\Big)\\
&\geq 2\gamma t_{k+1} - 2\gamma t_k - \gamma L(\|x_k - x_{k-1}\|^2 + \|x_{k+1} - x_k\|^2)\\
&\quad - \frac{\gamma L}{2}(\|x_{k-1} - x_{k-2}\|^2 + \|x_{k+1} - x_k\|^2) - \alpha(\|x_k - x_{k-1}\|^2 + \|x_{k+1} - x_k\|^2),
\end{aligned}$$

which is equivalent to

$$\begin{aligned}
&\|x_{k+1} - x\|^2 - \alpha\|x_k - x\|^2 + 2\gamma t_{k+1} + \big(1 - \alpha - \frac{3\gamma L}{2}\big)\|x_{k+1} - x_k\|^2\\
&\leq \|x_k - x\|^2 - \alpha\|x_{k-1} - x\|^2 + 2\gamma t_k + (2\alpha + \gamma L)\|x_k - x_{k-1}\|^2 + \frac{\gamma L}{2}\|x_{k-1} - x_{k-2}\|^2.
\end{aligned}$$

The proof is completed. □

The convergence of Algorithm 3.1 is presented in the following theorem.

Theorem 3.2. *Let $(x_k)_{k\in\mathbb{N}}$ be the sequence generated by Algorithm 3.1 with $\alpha \in [0, \frac{1}{3})$, and assume that*

$$\gamma < \frac{1 - 3\alpha}{3L}. \qquad (3.8)$$

Then $(x_k)_{k\in\mathbb{N}}$ converges weakly to $\overline{x} \in \operatorname{zer}(A + B)$.

Proof. For $x \in \operatorname{zer}(A+B)$, using (3.2), we obtain

$$\begin{aligned}
&\|x_{k+1}-x\|^2 - \alpha\|x_k - x\|^2 + 2\gamma t_{k+1} + (2\alpha + \frac{3\gamma L}{2})\|x_{k+1}-x_k\|^2 + \frac{\gamma L}{2}|\|x_k - x_{k-1}\|^2 \\
&\le \|x_k - x\|^2 - \alpha\|x_{k-1}-x\|^2 + 2\gamma t_k + (2\alpha + \frac{3\gamma L}{2})\|x_k - x_{k-1}\|^2 \\
&\quad + \frac{\gamma L}{2}\|x_{k-1}-x_{k-2}\|^2 - \big(1 - 3\alpha - 3\gamma L\big)\|x_{k+1}-x_k\|^2. \qquad (3.9)
\end{aligned}$$

We denote

$$S_k = \|x_k - x\|^2 - \alpha\|x_{k-1}-x\|^2 + 2\gamma t_k + (2\alpha + \frac{3\gamma L}{2})\|x_k - x_{k-1}\|^2 + \frac{\gamma L}{2}\|x_{k-1}-x_{k-2}\|^2,$$

we rewrite (3.9) as

$$S_{k+1} \le S_k - \big(1 - 3\alpha - 3\gamma L\big)\|x_{k+1}-x_k\|^2. \qquad (3.10)$$

We now prove that $(\forall k \in \mathbb{N})\ S_k \ge 0$. Indeed, by the formula of t_k, and by using Cauchy-Schwarz inequality and the Lipschitz property of B, we get

$$2|t_k| \le L(\|x_{k-1}-x_k\|^2 + \|x_k - x\|^2) + \frac{L}{2}(\|x_{k-1}-x_{k-2}\|^2 + \|x_k - x\|^2).$$

Hence,

$$\begin{aligned}
S_k &\ge \|x_k - x\|^2 - \alpha\|x_{k-1}-x\|^2 + (2\alpha + \frac{3\gamma L}{2})\|x_k - x_{k-1}\|^2 \\
&\quad - \gamma L(\|x_{k-1}-x_k\|^2 + \|x_k - x\|^2) - \frac{\gamma L}{2}\|x_k - x\|^2 \\
&\ge \big(1 - \frac{3\gamma L}{2}\big)\|x_k - x\|^2 - \alpha\|x_{k-1}-x\|^2 + 2\alpha\|x_k - x_{k-1}\|^2 \\
&= (1 - 2\alpha - \frac{3\gamma L}{2})\|x_k - x\|^2 + \alpha(2\|x_k - x\|^2 + 2\|x_k - x_{k-1}\|^2 - \|x_{k-1}-x\|^2) \\
&\ge (1 - 2\alpha - \frac{3\gamma L}{2})\|x_k - x\|^2 \ge 0. \qquad (3.11)
\end{aligned}$$

By combining (3.10), (3.11), and the condition (3.8), we get

$$\begin{cases} \lim_{k\to+\infty}\|x_{k+1}-x_k\| = 0 \\ \exists \lim_{k\to+\infty} S_k = \xi \in \mathbb{R}. \end{cases} \qquad (3.12)$$

It follows from (3.11) and (3.12) that the sequence $(x_k)_{k\in\mathbb{N}}$ is bounded and

$$\lim_{k\to\infty} S_k = \lim_{k\to\infty}(\|x_k - x\|^2 - \alpha\|x_{k-1}-x\|^2) = \xi.$$

Let $x^\star \in \mathcal{H}$ be a weak sequential cluster point of $(x_k)_{k\in\mathbb{N}}$. Then there exists a subsequence $(x_{k_n})_{n\in\mathbb{N}}$ that converges weakly to $x^\star$.
From (3.3), we have

$$\frac{x_k + \alpha(x_k - x_{k-1}) - x_{k+1}}{\gamma} - \big(\frac{5}{2}Bx_k - 2Bx_{k-1} + \frac{1}{2}Bx_{k-2}\big) + Bx_{k+1} \in (A+B)x_{k+1},$$

which is equivalent to

$$\begin{aligned}\frac{x_k - x_{k+1}}{\gamma} + \frac{\alpha(x_k - x_{k-1})}{\gamma} - (Bx_k - Bx_{k+1}) - \frac{3}{2}(Bx_k - Bx_{k-1})\\ + \frac{1}{2}(Bx_{k-1} - Bx_{k-2}) \in (A+B)x_{k+1}.\end{aligned} \tag{3.13}$$

Using the Lipschitz condition of B and $\lim_{k\to\infty}\|x_{k+1} - x_k\| = 0$, we see that the left-hand side of (3.13) converges strongly to 0. By the result of [11, Corollary 24.4], we can conclude that the sum $A + B$ is maximally monotone, and hence, its graph is closed in $\mathcal{H}^{weak} \times \mathcal{H}^{strong}$ [11, Proposition 20.33]. Therefore $x^\star \in \operatorname{zer}(A + B)$.

Assume that $(x_{k_n})_{n\in\mathbb{N}} \rightharpoonup x$, $(x_{l_n})_{n\in\mathbb{N}} \rightharpoonup y$. Then, we have

$$\begin{aligned}&-\big(\|x_k - x\|^2 - \alpha\|x_{k-1} - x\|^2\big) + \big(\|x_k - y\|^2 - \alpha\|x_{k-1} - y\|^2\big)\\ &+ (\|x\|^2 - \alpha\|x\|^2) - (\|y\|^2 - \alpha\|y\|^2)\\ &= 2\big(\langle x_k \mid x - y\rangle - \alpha\langle x_{k-1} \mid x - y\rangle\big).\end{aligned} \tag{3.14}$$

Choosing $k = k_n$ and $k = l_n$ then taking limit both sides of (3.14) when $n \to \infty$, we get

$$\|x - y\|^2 - \alpha\|x - y\|^2 = 0,$$

which implies $x = y$. Therefore $(x_k)_{k\in\mathbb{N}}$ converges weakly to a point in $\operatorname{zer}(A + B)$. The proof is completed. □

Next, we consider some simple examples to illustrate the effectiveness of our method. We compare our method to two methods: forward-backward-forward method (FBF) in [12] and the forward-reflected-backward method (FRB) in [14].

Example 1: We consider problem (1.1) with $\mathcal{H} = \mathbb{R}^n$, $Ax = 0$, $Bx = x$ and the initial values $x_0 \in \mathbb{R}^n$, $x_1 = -\frac{x_0}{2}$, $x_2 = \frac{x_0}{4}$ for all three methods. Note that 0 is the unique solution of (1.1) and B is 1−Lipschitz.
FBF method: $x_{k+1} = (1 - \gamma + \gamma^2)x_k$ for $\gamma < 1$. The optimal stepsize is $\gamma = \frac{1}{2}$ which gives a rate of $\frac{3}{4}$.
FRB method: $x_{k+1} = (1 - 2\gamma)x_k + \gamma x_{k-1}$ for $\gamma < \frac{1}{2}$. The optimal stepsize is $\gamma \approx \frac{1}{2}$ which gives a rate of $\frac{1}{\sqrt{2}}$.
Our method: We Choose $\alpha = \frac{1}{10}$, $\gamma = \frac{18}{85}$, then (3.1) becomes

$$x_{k+1} = \frac{97}{170}x_k + \frac{11}{34}x_{k-1} - \frac{9}{85}x_{k-2}.$$

The proposed method is $x_k = \dfrac{x_0}{(-2)^k}$ which gives a rate of $\frac{1}{2}$. We see that $\frac{1}{2} < \frac{1}{\sqrt{2}} < \frac{3}{4}$, therefore the proposed method in this paper converges faster FBF and FRB methods for this particular problem.

The convergence of the three methods are illustrated in Figure 1. Note that, in this case, FBF and FRB methods are optimally selected, i.e., the stepsize is equal $\frac{1}{2}$ is optimal.

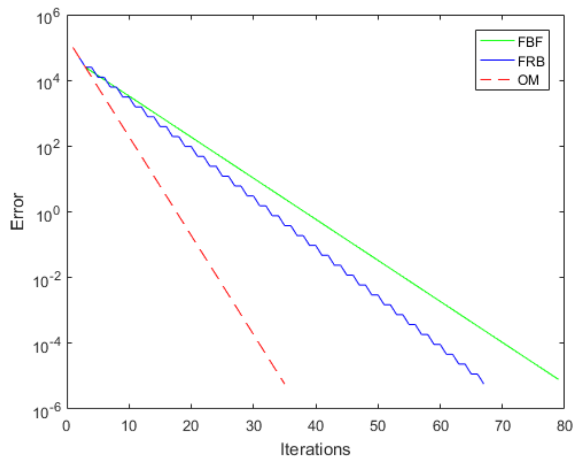

Figure 1. Convergence of the iteration of three methods

Example 2: Consider problem (1.1) with $\mathcal{H} = \mathbb{R}^2$, $A(z_1, z_2) = (0, 0)$, $B(z_1, z_2) = (z_2, -z_1)$. The convergence of FBF, FRB and Algorithm 3.1 are illustrated in Figures 2 and 3. In this case, we consider a common value of γ for three methods. We see that the convergence of our method is the same as FRB and faster than FBF method.

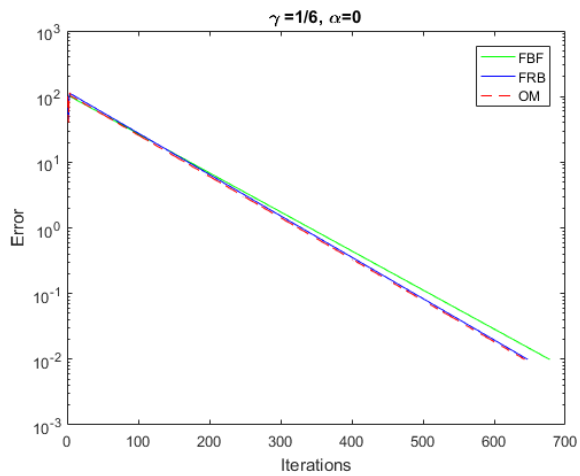

Figure 2. Convergence of the iteration of three methods

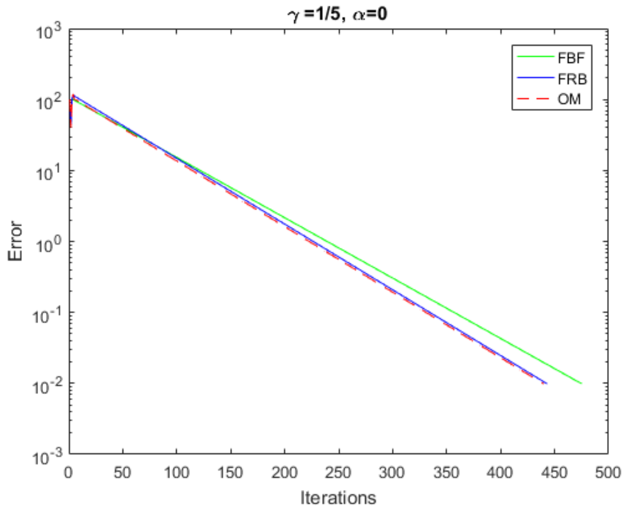

Figure 3. Convergence of the iteration of three methods

Example 3: We consider Example 2 again: $\mathcal{H} = \mathbb{R}^2$, $A(z_1, z_2) = (0, 0)$, $B(z_1, z_2) = (z_2, -z_1)$. The convergence of FBF, FRB and Algorithm 3.1 with the different initial values are illustrated in Figures 4 and 5. In this case, we choose $\alpha = 0$, $\gamma = 1/7$ for Algorithm 3.1 and $\gamma = 1/9$ for the other two methods. We see that the convergence of our method is faster than the two other methods. For all the other initial values, the convergence of three methods is the same as shown in Figures 4 and 5

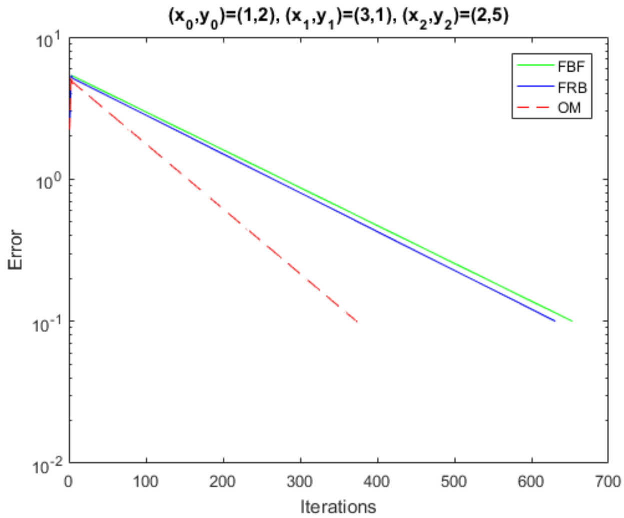

Figure 4. Convergence of the iteration of three methods

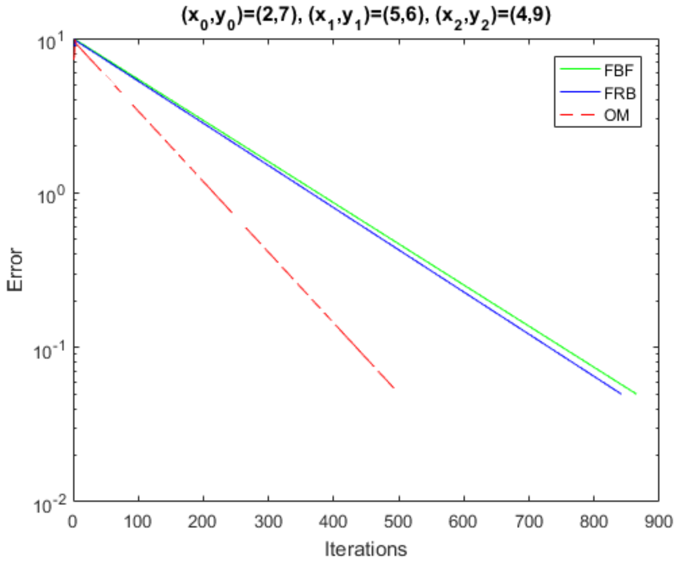

Figure 5. Convergence of the iteration of three methods

4. Conclusions

The paper has proposed an inertial splitting method for finding a zero point of the sum of a maximally monotone and a monotone-Lipschitz one. Under suitable conditions of the parameters, we have proved weak convergence of the algorithm. In some special cases, the proposed method converges faster than some known methods.

Acknowledgements. This research is funded by the University of Transport and Communications (UTC) under grant number T2025-CB-005.

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