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ROBUST STABILITY OF UNCERTAIN HOPFIELD NEURAL NETWORKS WITH PROPORTIONAL DELAYS

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Abstract. The problem of robust stability is investigated for a class of uncertain Hopfield-type neural networks with proportional delays. The existence and uniqueness of an equilibrium is first established using the homeomorphic mapping theorem. Then, by employing a modified Lyapunov–Krasovskii functional, a new criterion for the global asymptotic stability of an equilibrium point of the system is formulated.
Keywords: Robust stability, homeomorphism mapping, interval matrices.

1. Introduction

Neural networks models, including biology and artificial models, are widely used to described dynamics of various real-world phenomena. Applications of artificial neural networks models can be found, for example, in image realization and processing, time series forecasting, speech recognition, or pattern recognition for medical visualization aids [1]-[4]. In real-world applications of neural networks, the existence, uniqueness and long-term behavior, typically asymptotic stability, of a unique equilibrium [5] are essential aspects. Futhermore, due to many technical reasons such as the limit of switching speed of amplifiers or the signal processing transmission through layers, the implementation of neural networks is often encountered with time delays. The presence of delays usually makes the behavior of the system more complicated and unpredictable [6], [7]. Thus, over the past few decades, remarkable research attention has been devoted to the study of performance analysis and synthesis of neural networks with delays [8]-[11]

On the other hand, in electronically implemented neural networks, beside the affect of time-delay, the interconnection coefficients involved in neural systems are also unavoidably disturbed by external effects. Thus, the robust stability of neural networks against such perturbations must be examined [12]. There are different approaches to modeling neural networks with uncertainties of which the interval uncertainty is one of the most commonly used methods [13].

Different from existing works, in this paper, we consider the problem of robust stability of uncertain Hopfield neural networks with proportional delays. As discussed in the literature [14], proportional delays belong to a special class of unbounded delays, by which the analysis is much more challenging than bounded delay terms. First, by utilizing the homeomorphic mapping theorem in nonlinear analysis, tractable conditions for the existence and uniqueness of an equilibrium point (EP) are derived. Then, based on a type of modified Lyapunov-Krasovskii functionals, new criteria for the global asymptotic stability of a unique EP of the system are formulated.

Notation. $\mathbb{R}^n$ is the Euclidean space with the vector norm $\|x\| = \sqrt{\sum_{i=1}^n x_i^2}$, $\mathbb{R}^n_+ = \{x \in \mathbb{R}^n : x \succeq 0\}$, and $|x| = (|x_i|) \in \mathbb{R}^n_+$ for a vector $x = (x_i) \in \mathbb{R}^n$. For any vectors $x, y \in \mathbb{R}^n$, $x \preceq y$ if $x_i \leq y_i$ and $x \prec y$ if $x_i < y_i$ for all $i \in [n] := \{1, 2, \dots, n\}$. The absolute of a matrix $A = (a_{ij})_{n\times n}$ is denoted by $|A| = (|a_{ij}|)_{n\times n}$; A is nonnegative, $A \succeq 0$, if $a_{ij} \geq 0$ and A is positive, $A \succ 0$, if $a_{ij} > 0$ for all i, j. $\lambda_M(A^\top A)$ and $\lambda_m(A^\top A)$ denote the maximum and the minimum real part of eigenvalues of the matrix $A^\top A$, respectively. $\|A\|_2 = [\lambda_M(A^\top A)]^{1/2}$ denotes the spectra norm.

2. Preliminaries

Consider the following Hopfield-type neural system with heterogeneous proportional delays

$$\dot{x}_i(t) = -c_i x_i(t) + \sum_{j=1}^n a_{ij}\tilde{f}_j(x_j(t)) + \sum_{j=1}^n a^d_{ij}\tilde{f}_j(x_j(p_{ij}t)) + I_i,\ i \in [n],\ t \geq 1, \quad (2.1)$$

where n represents the number of neurons, $x_i(t)$ is the state of ith neuron at time t, c_i represents the charging rate of neuron ith, and I_i is external input. The system coefficients a_{ij}, a^d_{ij}, $i \in [n]$, are neural connection weights, $0 < p_{ij} < 1$ represent proportional delays according to $p_{ij}t = t - (1 - p_{ij})t$, $\tilde{f}_j(.)$, $j \in [n]$, are neural activation functions.

We assume that the connection weights c_i, a_{ij} and a^d_{ij} in system (2.1) are uncertain and bounded. More precisely, the system matrices are assumed to belong to the intervals

$$\begin{aligned}
&C_I := [\underline{C}, \overline{C}] = \{C = \mathrm{diag}\{c_i\} : 0 < \underline{c}_i \leq c_i \leq \bar{c}_i, i \in [n]\},\\
&A_I := [\underline{A}, \overline{A}] = \{A = (a_{ij})_{n\times n} : \underline{a}_{ij} \leq a_{ij} \leq \bar{a}_{ij}, i \in [n], j \in [n]\},\\
&A^d_I := [\underline{A}^d, \overline{A}^d] = \{A^d = (a^d_{ij})_{n\times n} : \underline{a}^d_{ij} \leq a^d_{ij} \leq \bar{a}^d_{ij}, i \in [n], j \in [n]\},
\end{aligned} \quad (2.2)$$

Assumption (A1): The neuron activation functions $\tilde{f}_j(.)$, $j \in [n]$, are continuous and there exist constants l^-_{jf}, l^+_{jf} that satisfy the following condition

$$l^-_{jf} \leq \frac{\tilde{f}_j(a) - \tilde{f}_j(b)}{a - b} \leq l^+_{jf}, \quad \forall a \neq b. \quad (2.3)$$

Remark 2.1. *By Assumption*(A1), the functions $\tilde{f}(x) = (\tilde{f}_j(x_j))^\top$, $x = (x_j)^\top$, $j \in [n]$, satisfies the following inequalities

$$|\tilde{f}_j(a) - \tilde{f}_j(b)| \leq F_j|a - b| \text{ for all } a, b \in \mathbb{R}, a \neq b,$$

where $F_j = \max\{l_{jf}^+, -l_{jf}^-\}$. Hereafter, we denote the matrix $F = \mathrm{diag}\{F_j\}$.

Definition 2.1. *A point $x^* \in \mathbb{R}^n$ is said to be an EP of system* (2.1) *if it holds that*

$$-Cx^* + A\tilde{f}(x^*) + A^d\tilde{f}(x^*) + I = 0. \tag{2.4}$$

Definition 2.2. *System* (2.1) *with uncertain matrices defined by* (2.2) *is said to be globally asymptotically robust stable if the unique EP $x^* \in \mathbb{R}^n$ of* (2.1) *is GAS (globally asymptotically stable) for all $C \in C_I$, $A \in A_I$, and $A^d \in A_I^d$.*

The following technical lemmas will be useful for our next derivation.

Lemma 2.1. *For any $x, z \in \mathbb{R}^n$ and positive scalar ϵ, the following inequality holds*

$$2x^\top z \leq \epsilon x^\top x + \epsilon^{-1} z^\top z.$$

Lemma 2.2. *Let $A = (a_{ij})_{n\times n} \in A_I$. For any positive matrices $M = \mathrm{diag}\{m_i\}$, $i \in [n]$, scalar $\alpha > 0$, and vectors $u = (u_i)$ and $v = (v_j)$ in $\mathbb{R}^n$, the following inequality holds*

$$\begin{aligned} 2u^\top MAv &\leq \alpha m^* \sum_{i=1}^n u_i^2 + \alpha^{-1} m^* \sum_{j=1}^n h_j v_j^2 \\ &= m^*\Big(\alpha u^\top u + \alpha^{-1} v^\top H v\Big), \end{aligned} \tag{2.5}$$

where $m^* = \max_{i\in[n]}\{m_i\}$, $H = \mathrm{diag}\{h_j\}$, $h_j = \sum_{i=1}^n \left(\hat{a}_{ij} \sum_{l=1}^n \hat{a}_{il}\right)$, $j \in [n]$, with $\hat{a}_{ij} = \max\{|\underline{a}_{ij}|, |\overline{a}_{ij}|\}$.

Proof. For a matrix $M = \mathrm{diag}\{m_i\} \succ 0$, we have

$$\begin{aligned} 2u^\top MAv &\leq 2|u^\top||M||A||v| = 2\sum_{i=1}^n \sum_{j=1}^n m_i |a_{ij}||u_i||v_j| \\ &\leq 2m^* \sum_{i=1}^n \sum_{j=1}^n |a_{ij}||u_i||v_j| = 2m^*|u^\top||A||v|. \end{aligned}$$

By Lemma 2.1,

$$2|u^\top||A||v| \leq \alpha u^\top u + \alpha^{-1}|v^\top||A^\top||A||v|,$$

and hence

$$\begin{aligned}
|v^\top||A^\top||A||v| &= \sum_{j=1}^{n}\left(\sum_{i=1}^{n}|a_{ij}||a_{ij}|\right)v_j^2 + \sum_{j=1}^{n}\sum_{l=j+1}^{n}\left(\sum_{i=1}^{n}2|a_{ij}||a_{il}||v_j||v_l|\right)\\
&\le \sum_{j=1}^{n}\left(\sum_{i=1}^{n}|a_{ij}||a_{ij}|\right)v_j^2 + \sum_{j=1}^{n}\sum_{l=j+1}^{n}\left(\sum_{i=1}^{n}|a_{ij}||a_{il}|(v_j^2+v_l^2)\right)\\
&= \sum_{j=1}^{n}\Big(\sum_{i=1}^{n}|a_{ij}||a_{ij}|\Big)v_j^2 + \sum_{j=1}^{n}\Big(\sum_{i=1}^{n}|a_{ij}|\sum_{l=1,l\neq j}^{n}|a_{il}|\Big)v_j^2\\
&= \sum_{j=1}^{n}\Big(\sum_{i=1}^{n}|a_{ij}|\sum_{l=1}^{n}|a_{il}|\Big)v_j^2\\
&\le \sum_{j=1}^{n}\Big(\sum_{i=1}^{n}\hat{a}_{ij}\sum_{l=1}^{n}\hat{a}_{il}\Big)v_j^2 = \sum_{j=1}^{n}h_jv_j^2 = v^\top Hv.
\end{aligned}$$

The last inequality shows that

$$2u^\top MAv \le m^*\left(\alpha u^\top u + \alpha^{-1}v^\top Hv\right)$$

as desired. □

In the remaining of this section, we recall an additional auxiliary result, which will be used to derive existence conditions. A mapping $\mathcal{F}:\mathbb{R}^n\to\mathbb{R}^n$ is said to be *proper* if the pre-image $\mathcal{F}^{-1}(K)$ is compact for any compact $K\subset\mathbb{R}^n$. It is clear that a continuous mapping $\mathcal{F}:\mathbb{R}^n\to\mathbb{R}^n$ is proper if and only if, for any sequence $\{p_k\}\subset\mathbb{R}^n$, $\|p_k\|\to\infty$, it holds that $\|\mathcal{F}(p_k)\|\to\infty$ as $k\to\infty$.

Lemma 2.3. *(see [15]) A locally invertible continuous mapping $\mathcal{F}:\mathbb{R}^n\to\mathbb{R}^n$ is a homeomorphism of $\mathbb{R}^n$ onto itself if and only if it is proper.*

3. Main results

3.1. Equilibrium

To facilitate in presenting our next results, we denote the matrix

$$\tilde{A}^d = \mathrm{diag}\{\tilde{a}_i^d\},\ \text{where } \tilde{a}_i^d = \sum_{j=1}^{n}\max\{|\underline{a}_{ij}^d|^2, |\overline{a}_{ij}^d|^2\},\ i\in[n].$$

Theorem 3.1. *Let Assumption* (A1) *hold and assume that there exist positive scalars α, γ, and a positive matrix $M=\mathrm{diag}\{m_i\}\succ 0$ such that*

$$\Theta := 2M\underline{C} - \alpha m^*E_n - \gamma M^2\tilde{A}^d - \alpha^{-1}m^*HF^2 - \gamma^{-1}nF^2 \succ 0, \tag{3.1}$$

where $E_n\in\mathbb{R}^{n\times n}$ is the identity matrix. Then, system (2.1) *possesses a unique EP $x^*\in\mathbb{R}^n$.*

Proof. We define a continuous mapping $\mathcal{F} : \mathbb{R}^n \to \mathbb{R}^n$ as

$$\mathcal{F}(u) = -Cu + A\tilde{f}(u) + A^d\tilde{f}(u) + I.$$

It is clear that $x^* \in \mathbb{R}^n$ is an EP of system (2.1) if and only if it is a null point of the mapping $\mathcal{F}$, that is, $\mathcal{F}(x^*) = 0$. We now show that, under the derived condition of Theorem 3.1, the mapping $\mathcal{F}(u)$ is proper. Thanks to Lemma 2.3, it suffices to prove that $\mathcal{F}(u)$ is a homeomorphism onto $\mathbb{R}^n$. Indeed, for any $u, v \in \mathbb{R}^n$, $u \neq v$, we have

$$\mathcal{F}(u) - \mathcal{F}(v) = -C(u - v) + A\big(\tilde{f}(u) - \tilde{f}(v)\big) + A^d\big(\tilde{f}(u) - \tilde{f}(v)\big).$$

Therefore,

$$\begin{aligned} 2(u-v)^\top M\big[\mathcal{F}(u) - \mathcal{F}(v)\big] &= -2(u-v)^\top MC(u-v) \\ &\quad + 2(u-v)^\top MA\big(\tilde{f}(u) - \tilde{f}(v)\big) \\ &\quad + 2(u-v)^\top MA^d\big(\tilde{f}(u) - \tilde{f}(v). \\ &\leq -2(u-v)^\top M\underline{C}(u-v) + 2(u-v)^\top MA\big(\tilde{f}(u) - \tilde{f}(v)\big) \\ &\quad + 2(u-v)^\top MA^d\big(\tilde{f}(u) - \tilde{f}(v). \end{aligned} \tag{3.2}$$

In addition, by Lemma 2.2, we have

$$\begin{aligned} 2(u-v)^\top MA\big(\tilde{f}(u) - \tilde{f}(v)\big) &\leq \alpha m^*(u-v)^\top(u-v) \\ &+ \alpha^{-1}m^*(\tilde{f}(u) - \tilde{f}(v))^\top H(\tilde{f}(u) - \tilde{f}(v)). \end{aligned} \tag{3.3}$$

Taking Assumption (A1) into account, we obtain

$$\begin{aligned} 2(u-v)^\top MA\big(\tilde{f}(u) - \tilde{f}(v)\big) &\leq \alpha m^*(u-v)^\top(u-v) \\ &+ \alpha^{-1}m^*(u-v)^\top HF^2(u-v). \end{aligned} \tag{3.4}$$

and

$$\begin{aligned}
2(u-v)^\top MA^d\big(\tilde f(u)-\tilde f(v)\big) &= 2\sum_{i=1}^n\sum_{j=1}^n (u_i-v_i)m_i a^d_{ij}\big(\tilde f_j(u_j)-\tilde f_j(v_j)\big)\\
&\le \sum_{i=1}^n\sum_{j=1}^n 2|u_i-v_i|m_i|a^d_{ij}||\tilde f_j(u_j)-\tilde f_j(v_j)|\\
&\le \sum_{i=1}^n\sum_{j=1}^n 2|u_i-v_i|m_i|a^d_{ij}|F_j|u_j-v_j|\\
&\le \gamma\sum_{i=1}^n\sum_{j=1}^n |u_i-v_i|^2|m_i^2|a^d_{ij}|^2+\gamma^{-1}\sum_{i=1}^n\sum_{j=1}^n F_j^2|u_j-v_j|^2\\
&= \gamma\sum_{i=1}^n m_i^2\big(\sum_{j=1}^n |a^d_{ij}|^2\big)|u_i-v_i|^2+\gamma^{-1}n\sum_{j=1}^n F_j^2|u_j-v_j|^2\\
&\le \gamma\sum_{i=1}^n m_i^2\tilde a_i^d|u_i-v_i|^2+\gamma^{-1}n\sum_{j=1}^n F_j^2|u_j-v_j|^2.
\end{aligned}$$

Thus,

$$2(u-v)^\top MA^d\big(\tilde f(u)-\tilde f(v)\big) \le \gamma(u-v)^\top M^2\tilde A^d(u-v)+\gamma^{-1}n(u-v)^\top F^2(u-v). \quad (3.5)$$

Combining (3.3), (3.4), and (3.5), we readily obtain

$$\begin{aligned}
2(u-v)^\top M\left[\mathcal F(u)-\mathcal F(v)\right] &\le (u-v)^\top\left[-2M\underline{C}+\alpha m^*E_n+\gamma M^2\tilde A^d\right](u-v)\\
&\quad+(u-v)^\top\left[\alpha^{-1}m^*HF^2+\gamma^{-1}nF^2\right](u-v). \quad (3.6)
\end{aligned}$$

Thus, by condition (3.1),

$$2(u-v)^\top M\Big[\mathcal F(u)-\mathcal F(v)\Big] \le -(u-v)^\top\Theta(u-v)<0. \quad (3.7)$$

The result of (3.7) induces that $\mathcal F(u)\ne\mathcal F(v)$ for all $u\ne v$. Thus, $\mathcal F(.)$ is an injective mapping in $\mathbb R^n$. In addition, it also follows from (3.7) that

$$2(u-v)^\top M\left[\mathcal F(u)-\mathcal F(v)\right] \le -\lambda_m(\Theta)\|u-v\|_2^2<0. \quad (3.8)$$

Let $v=0$, we have

$$2u^\top M\left[\mathcal F(u)-\mathcal F(0)\right]\le -\lambda_m(\Theta)\|u\|_2^2<0$$

and, therefore,

$$\|\mathcal F(u)-\mathcal F(0)\|_2 \ge \frac{\lambda_m(\Theta)\|u\|_2}{2m^*}.$$

By using the property $\|\mathcal{F}(u) - \mathcal{F}(0)\|_2 \leq \|\mathcal{F}(u)\|_2 + \|\mathcal{F}(0)\|_2$, we finally obtain

$$\|\mathcal{F}(u)\|_2 \geq \frac{\lambda_m(\Theta)}{2m^*}\|u\|_2 - \|\mathcal{F}(0)\|_2$$

which ensures that $\|\mathcal{F}(p_k)\|_2 \to \infty$ for any sequence $\{p_k\} \subset \mathbb{R}^n$ with $\|p_k\|_2 \to \infty$ as $k \to \infty$. Thus, $\mathcal{F}(.)$ is a homeomorphism onto $\mathbb{R}^n$. Consequently, $\mathcal{F}(u) = 0$ has a unique solution $x^* \in \mathbb{R}^n$, which is a unique EP of (2.1). The proof is completed. □

3.2. Robust stability

Based on the result of Theorem 3.1, in this section, we prove the property of global asymptotic stability of the EP x^* of system (2.1). For this, we define a state transformation as

$$u_i(t) = x_i(e^t) - x_i^*,\ i \in [n].$$

Then, system (2.1) can be recast to the following system with constant delays and variable coefficients

$$\dot{u}_i(t) = e^t \left(-c_i u_i(t) + \sum_{j=1}^{n} a_{ij} f_j(u_j(t)) + \sum_{j=1}^{n} a_{ij}^d f_j(u_j(t - \tau_{ij})) \right),\ t \geq 0, \tag{3.9}$$

where $\tau_{ij} = -\ln p_{ij} > 0$ and $f_j(u_j(t)) = \tilde{f}_j(u_j(t) + x_j^*) - \tilde{f}_j(x_j^*),\ j \in [n]$.

By (A1), the function f_j satisfies $f_j(0) = 0$ and

$$|f_j(u_j(t))| \leq F_j |u_j(t)|. \tag{3.10}$$

In addition, it can be verified that the EP $x^* \in \mathbb{R}^n$ of system (2.1) is shifted to the origin of system (3.9). Therefore, the EP x^* of (2.1) is GAS if the origin of system (3.9) is GAS.

Theorem 3.2. *Under the assumptions of Theorem 3.1, that is, there exist positive scalars α, γ and a positive matrix $M = \mathrm{diag}\{m_i\} \succ 0$ satisfying condition (3.1), system (2.1) possesses a unique EP $x^* \in \mathbb{R}^n$ which is robust globally asymptotically stable.*

Proof. Consider the following Lyapunov-like functional

$$V(u(t)) = e^{-t} \sum_{i=1}^{n} m_i u_i^2(t) + \gamma^{-1} \sum_{i=1}^{n} \sum_{l=1}^{n} \int_{t-\tau_{il}}^{t} f_l^2(u_l(\xi)) d\xi. \tag{3.11}$$

The derivative of $V(u(t))$ along state trajectories of system (3.9) is given as

$$
\begin{aligned}
\dot{V}(u(t)) &= -e^{-t}\sum_{i=1}^{n} m_i u_i^2(t) + 2e^{-t}\sum_{i=1}^{n} u_i(t)\dot{u}_i(t) \\
&\quad + \gamma^{-1}\sum_{i=1}^{n}\sum_{l=1}^{n}\Big(f_l^2(u_l(t)) - f_l^2(u_l(t-\tau_{il}))\Big) \\
&\le 2e^{-t}\sum_{i=1}^{n} u_i(t)\dot{u}_i(t) + \gamma^{-1}\sum_{i=1}^{n}\sum_{l=1}^{n}\Big(f_l^2(u_l(t)) - f_l^2(u_l(t-\tau_{il}))\Big) \\
&= 2\sum_{i=1}^{n} u_i(t)\Big(-c_i u_i(t) + \sum_{j=1}^{n} a_{ij} f_j(u_j(t)) + \sum_{j=1}^{n} a_{ij}^d f_j(u_j(t-\tau_{ij}))\Big) \\
&\quad + \gamma^{-1}\sum_{i=1}^{n}\sum_{l=1}^{n}\Big(f_l^2(u_l(t)) - f_l^2(u_l(t-\tau_{il}))\Big) \\
&= -2\sum_{i=1}^{n} m_i c_i u_i^2(t) + 2\sum_{i=1}^{n}\sum_{j=1}^{n} m_i u_i(t) a_{ij} f_j(u_j(t)) \\
&\quad + 2\sum_{i=1}^{n}\sum_{j=1}^{n} m_i u_i(t) a_{ij}^d f_j(u_j(t-\tau_{ij})) \\
&\quad + \gamma^{-1}\sum_{i=1}^{n}\sum_{l=1}^{n}\Big(f_l^2(u_l(t)) - f_l^2(u_l(t-\tau_{il}))\Big) \\
&\le -2\sum_{i=1}^{n} m_i\ \underline{c_i}\ u_i^2(t) + 2\sum_{i=1}^{n}\sum_{j=1}^{n} m_i u_i(t) a_{ij} f_j(u_j(t)) \\
&\quad + 2\sum_{i=1}^{n}\sum_{j=1}^{n} m_i u_i(t) a_{ij}^d f_j(u_j(t-\tau_{ij})) \\
&\quad + \gamma^{-1} n\sum_{l=1}^{n} F_l^2 |u_l(t)|^2 - \gamma^{-1}\sum_{i=1}^{n}\sum_{l=1}^{n} f_l^2(u_l(t-\tau_{il}))\Big).
\end{aligned}
$$

By Lemma 2.2 and condition (3.10), we have

$$
\begin{aligned}
&2\sum_{i=1}^{n}\sum_{j=1}^{n} m_i u_i(t) a_{ij} f_j(u_j(t))) = 2u^\top(t) M A f(u(t)) \\
&\le \alpha m^* u^\top(t) u(t) + \alpha^{-1} m^* u^\top(t) H F^2 u(t). \qquad (3.12)
\end{aligned}
$$

Next, we estimate the term $2\sum_{i=1}^{n}\sum_{j=1}^{n} m_i u_i(t) a_{ij}^d f_j(u_j(t-\tau_{ij}))$ as follows

$$
\begin{aligned}
2\sum_{i=1}^{n}\sum_{j=1}^{n} m_i u_i(t) a_{ij}^d f_j(u_j(t-\tau_{ij})) &\leq \sum_{i=1}^{n}\sum_{j=1}^{n} 2m_i|u_i(t)||a_{ij}^d||f_j(u_j(t-\tau_{ij})| \\
&\leq \gamma \sum_{i=1}^{n}\sum_{j=1}^{n} m_i^2 u_i^2(t)|a_{ij}^d|^2 + \gamma^{-1}\sum_{i=1}^{n}\sum_{j=1}^{n} f_j^2(u_j(t-\tau_{ij}) \\
&\leq \gamma u^\top(t) M^2 \tilde{A}^d u(t) + \gamma^{-1}\sum_{i=1}^{n}\sum_{j=1}^{n} f_j^2(u_j(t-\tau_{ij}). \qquad (3.13)
\end{aligned}
$$

Combining (3.12) and (3.13), the derivative of $V(u(t))$ can be manipulated as

$$
\begin{aligned}
\dot{V}(u(t))) &\leq -u^\top(t)\left[2M\underline{C} - m^*\alpha E_n - \gamma M^2\tilde{A}^d - m^*\alpha^{-1}HF^2 - \gamma^{-1}nF^2\right]u(t) \\
&\leq -u^\top(t)\Theta u(t) \\
&\leq -\lambda_m(\Theta)\|u(t)\|_2^2,
\end{aligned} \qquad (3.14)
$$

where the matrix Θ is formulated in Theorem 3.1. According to condition (3.1), $\lambda_m(\Theta) > 0$, we have

$$\dot{V}(u(t)) < 0, \ \forall u(t) \neq 0.$$

By this negativeness condition, it can be concluded that the origin of system (3.9) is globally asymptotically stable. The proof is completed. □

Remark 3.1. *As a special case, if the neuron connection weights of model* (2.1) *are known (i.e. the matrices C, A and A^d are known), the result of Theorem 3.2 is reduced to the following corollary.*

Corollary 3.1. *Let Assumption* (A1) *hold and assume that there exist positive scalars α, γ, and a positive matrices $M = \mathrm{diag}\{m_i\} \succ 0$ that satisfy the following condition*

$$\tilde{\Theta} = 2MC - \alpha m^* E_n - \gamma M^2\hat{A}^d - \alpha^{-1}m^*\tilde{H}F^2 - \gamma^{-1}nF^2 \succ 0, \qquad (3.15)$$

where $\hat{A}^d = \mathrm{diag}\{\hat{a}_i^d\}$, $\tilde{H} = \mathrm{diag}\{\tilde{h}_j\}$, and $\hat{a}_i^d = \sum_{j=1}^{n}|a_{ij}^d|^2$, $\tilde{h}_j = \sum_{i=1}^{n}(|a_{ij}|\sum_{l=1}^{n}|a_{il}|)$. Then, system (2.1) *has a unique EP $x^* \in \mathbb{R}^n$ which is GAS.*

Remark 3.2. *The results of Theorems 3.1, 3.2 can be extended for the following uncertain neural networks model*

$$\dot{x}_i(t) = -c_i x_i(t) + \sum_{j=1}^{n} a_{ij}\tilde{f}_j^{(1)}(x_j(t)) + \sum_{j=1}^{n} a_{ij}^d \tilde{f}_j^{(2)}(x_j(p_{ij}t)) + I_i, \ i \in [n], \qquad (3.16)$$

where the neural activation functions $\tilde{f}_j^{(1)}$ and $\tilde{f}_j^{(2)}$, $j \in [n]$, satisfy the following conditions

$$
\begin{aligned}
|\tilde{f}_j^{(1)}(a) - \tilde{f}_j^{(1)}(b)| &\leq F_j^{(1)}|a-b|, \\
|\tilde{f}_j^{(2)}(a) - \tilde{f}_j^{(2)}(b)| &\leq F_j^{(2)}|a-b|
\end{aligned} \qquad (3.17)
$$

for all $a, b \in \mathbb{R}$, $a \neq b$. *Denote* $F_1 = \operatorname{diag}\{F_j^{(1)}\}$ *and* $F_2 = \operatorname{diag}\{F_j^{(2)}\}$. *We have the following result.*

Theorem 3.3. *Consider system* (3.16) *with uncertain parameters defined by* (2.2). *Assume that condition* (3.17) *holds and there exist positive constants* α, γ *and positive matrix* $M = \operatorname{diag}\{m_i\} \succ 0$ *that satisfy the following condition*

$$\Theta_1 = 2M\underline{C} - \alpha m^* E_n - \gamma M^2 \tilde{A}^d - \alpha^{-1} m^* H F_1^2 - \gamma^{-1} n(F_1^2 + F_2^2) \succ 0. \qquad (3.18)$$

Then, system (3.16) *possesses a unique EP* $x^* \in \mathbb{R}^n$, *which is GAS.*

Proof. The existence and uniqueness of an EP of system (3.16) can be demonstrated by similar arguments used in the proof of Theorem 3.1. To prove the EP of system (3.16) is globally asymptotically stable, we consider the following Lyapunov-like functional

$$\begin{aligned} V(u(t)) &= e^{-t}\sum_{i=1}^{n} m_i u_i^2(t) \\ &+ \gamma^{-1}\sum_{i=1}^{n}\sum_{l=1}^{n}\left(\int_{t-\tau_{il}}^{t} (f_l^{(1)})^2(u_l(\xi))d\xi + \int_{t-\tau_{il}}^{t} (f_l^{(2)})^2(u_l(\xi))d\xi\right), \end{aligned} \qquad (3.19)$$

where

$$\begin{aligned} f_l^{(1)}(u_l(t)) &= \tilde{f}_l^{(1)}(u_l(t) + x_l^*) - \tilde{f}_l^{(1)}(x_l^*), \\ f_l^{(2)}(u_l(t-\tau_{il})) &= \tilde{f}_l^{(2)}(u_l(t-\tau_{il}) + x_l^*) - \tilde{f}_l^{(2)}(x_l^*),\ l \in [n],\ i \in [n]. \end{aligned}$$

By similar arguments used in the proof of Theorem 3.2, it can be shown under condition (3.18) that the derivative of the functional (3.19) is negative definite. Thus, the EP x^* of system (3.18) is GAS. The proof is completed. □

4. Numerical example

Consider system (2.1), where the neuron connection weights satisfy (2.2) with the data

$$\begin{aligned} \underline{A} &= -1_{4\times 4}, \quad \overline{A} = 1_{4\times 4}, \\ \underline{A^d} &= -2.1_{4\times 4}, \quad \overline{A^d} = 2.1_{4\times 4} \end{aligned}$$

and $\underline{C} = \overline{C} = \operatorname{diag}\{c_1, c_2, c_3, c_4\}$, where $1_{m\times n}$ denotes the (m,n)-matrix with all elements equal one. The activation functions are given by

$$\begin{aligned} \tilde{f}_1(x_1) &= \tanh(L_1 x_1), \quad \tilde{f}_2(x_2) = \tanh(L_2 x_2), \\ \tilde{f}_3(x_3) &= \sin(L_3 x_3), \quad \tilde{f}_4(x_4) = \sin(L_4 x_4), \end{aligned}$$

where $L_1 = L_2 = \frac{1}{4}$ and $L_3 = L_4 = \frac{1}{2}$. It is clear that the functions $\tilde{f}_j$ are continuous, differentiable on $\mathbb{R}$ and satisfy $|\tilde{f}_j'(x_j)| \le L_j$, $j = 1, 2, 3, 4$. Thus, by mean valued theorem, we can easily see that Assumption (A1) holds with $F = \mathrm{diag}\{\frac{1}{4}, \frac{1}{4}, \frac{1}{2}, \frac{1}{2}\}$. In addtion, by a simple calculation, we have $\tilde{A}^d = H = 16E_4$. We choose $\alpha = 1$, $\gamma = \frac{1}{16}$ and $M = \mathrm{diag}\{1, 2, 2, 2\}$, then

$$\Theta = \mathrm{diag}\{2c_1 - 9, 4c_2 - 12, 4c_3 - 30, 4c_4 - 30\}.$$

It is clear that $\Theta \succ 0$ if and only if $c_1 > 4.5$, $c_2 > 3$, $c_3 > 7.5$ and $c_4 > 7.5$. By Theorems 3.1 and 3.2, system (2.1) has a unique EP that is globally asymptotically stable.

5. Conclusions

The problem of robust stability of uncertain Hopfield neural networks with proportional delays has been investigated in this paper. The existence and uniqueness of an equilibrium have been established using the homeomorphic mapping theorem. By utilizing appropriate Lyapunov-like functionals, new criteria have been established to determine the global asymptotic stability of the unique equilibrium point. Finally, a numerical example has been provided to demonstrate the effectiveness of the obtained results.

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