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ON UNIQUENESS OF MEROMORPHIC FUNCTIONS WITH FINITE GROWTH INDEX SHARING SOME SMALL FUNCTIONS

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Abstract. In this paper, we will prove a uniqueness theorem for meromorphic functions with finite growth indices on a complex disc sharing some small functions with different multiplicity values. Intersecting points between these mappings and small functions with multiplicities more than a certain number do not need to be counted. Our result extends some previous results on this topic.
Keywords: meromorphic function, unicity, complex disc.

1. Introduction

From the theorems about the four and five values of Nevanlinna R [1], many authors have improved and generalized these theorems to prove the finiteness problem of meromorphic mappings on $\mathbb{C}^m$, a Kähler manifold, a semi-Abelian variety or an annuli, etc. We can see these results in [2]-[6]. In 2020, Ru M and Sibony N [7] formulated a new second main theorem for meromorphic functions on a complex disc with fixed values, and then in 2022, Si DQ [8] generalized that result by using small functions instead of fixed values. In this paper, he also proved an uniqueness theorem for non-constant meromorphic functions on a disc with finite growth indices sharing small functions as follows:

Theorem A *Let f, g be two non-constant meromorphic functions on the disc $\Delta(R)$ $(0 < R \leq +\infty)$ with finite growth indices c_f, c_g. Let $\{(a_i)\}_{i=1}^q$ $(q \geq 5)$ be q distinct small functions (with respect to f and g) and k be a positive integers or $+\infty$. Assume that*

$$\min\{1, \nu^0_{f-a_i,\leq k}\} = \min\{1, \nu^0_{g-b_i,\leq k}\}\ (1 \leq i \leq q).$$

If $c_f + c_g < \dfrac{k(2q-8) - 3(q+4)}{k(19q - \frac{117}{2}) + 19(q+4)}$ *then* $f \equiv g$.

However, S. D. Quang only considered the case where the mappings f and g share q $(q \geq 5)$ small functions in $\Delta(R)$ which have the same multiplicities. The purpose of this paper is to improve the result of Theorem A by giving a unique theorem with differentmultiplicity values. Specifically, we will prove the following theorem.

Theorem 1.1. *Let f, g be two non-constant meromorphic functions on the disc $\Delta(R)$ $(0 < R \leq +\infty)$ with finite growth indices c_f, c_g. Let $\{(a_i)\}_{i=1}^q$ $(q \geq 5)$ be q distinct small functions (with respect to f and g) and $k_1, ..., k_q$ be a positive integers or ∞ such that*

$$c_f + c_g < \frac{qk_0(2q-8) + 2q(q+4) - 5(q+4)\sum_{i=1}^q \frac{1}{k_i}}{qk_0(19q - \frac{117}{2}) + 19q(q+4)}.$$

where $k_0 = \max_{1 \leq i \leq q} k_i$. Assume that

$$\min\{1, \nu^0_{f-a_i, \leq k_i}\} = \min\{1, \nu^0_{g-b_i, \leq k_i}\} \ (1 \leq i \leq q).$$

Then $f \equiv g$.

Remark. When $k_1 = k_2 = \cdots = k_q = k$, from Theorem 1.1, we obtain the result of the Theorem A.

2. Some results from Nevanlinna theory on the complex disc

Now, we set a disc in $\mathbb{C}$ by

$$\Delta(R) = \{z \in \mathbb{C} : |z| < R\} \ (0 < R \leq +\infty).$$

For a divisor ν on $\Delta(R)$, which can be regarded as a function on $\Delta(R)$ with value in $\mathbb{Z}$ whose support is a discrete subset of $\Delta(R)$, and for a positive integer M (maybe $M = \infty$), we define the truncated counting function to level M of ν by

$$n^{[M]}(t, \nu) = \sum_{|z_\nu| \leq t} \min\{M, \nu(z)\} \ (0 \leq t \leq R),$$

$$\text{and } N^{[M]}(r, \nu) = \int_0^r \frac{n^{[M]}(t,\nu) - n^{[M]}(0,\nu)}{t} dt.$$

For brevity we will omit the character $^{[M]}$ if $M = +\infty$.

For a divisor ν and a positive integer k (maybe $k = +\infty$), we define

$$\nu_{\leq k}(z) = \begin{cases} \nu(z) & \text{if } \nu(z) \leq k \\ 0 & \text{otherwise} \end{cases} \text{ and } \nu_{>k}(z) = \begin{cases} \nu(z) & \text{if } \nu(z) > k \\ 0 & \text{otherwise.} \end{cases}$$

For a meromorphic function φ, we define

- ν^0_φ (resp. ν^∞_φ) the divisor of zeros (resp. divisor of poles) of φ,
- $\nu_\varphi = \nu^0_\varphi - \nu^\infty_\varphi$,
- $\nu^0_{\varphi,\leq k} = (\nu^0_\varphi)_{\leq k}, \nu^0_{\varphi,>k} = (\nu^0_\varphi)_{>k}$.

Similarly, we define $\nu^\infty_{\varphi,\leq k}$, $\nu^\infty_{\varphi,\leq k}$, $\nu_{\varphi,\leq k}$, $\nu_{\varphi,\leq k}$ and their counting functions.

Let f be a nonconstant meromorphic function on $\Delta(R)$. We define the proximity function and the characteristic function of f as follows:

$$m(r,f) = \frac{1}{2\pi}\int\limits_0^{2\pi} \log^+ |f(re^{i\theta})| d\theta,$$

and

$$T(r,f) = m(r,f) + N(r,\nu^\infty_f).$$

A meromorphic function a is said to be small with respect to f if $T(r,a) = o(T(r,f))$ as $r \to R$.

According to M. Ru and N. Sibony [7], the growth index of f is defined by

$$c_f = \inf\{c > 0 : \int_0^R e^{cT(r,f)} dr = +\infty\}.$$

For convenient, we will set $c_f = +\infty$ if $\{c > 0 : \int_0^R e^{cT(r,f)} dr = +\infty\} = \emptyset$.

For given two meromorphic mappings f and g on $\Delta(R)$ (here, we may use a conformal transformation from a plane to a disc), the map f is said to be a quasi-Möbius transformation of g if there exist small (with respect to g) functions α_i $(1 \leq i \leq 4)$ such that $f = \frac{\alpha_1 g + \alpha_2}{\alpha_3 g + \alpha_4}$. If all functions α_i $(1 \leq i \leq 4)$ are constants then we say that the map f is a Möbius transformation of g.

Throughout this paper, by notation "$\|_E\ P$", we mean that the asseartion P hold for all $r \in (0,R)$ outside a subset E of $(0,R)$ with $\int_E \gamma(r)dr < +\infty$.

Lemma 2.1 (Lemma on logarithmic derivatives [7])**.** *Let $0 < R \leq +\infty$ and let $\gamma(r)$ be a non-negative measurable function defined on $(0,R)$ with $\int_0^R \gamma(r)dr = +\infty$. Let f be a nonzero meromorphic function on $\Delta(R)$. Then for $\varepsilon > 0$, we have*

$$\|_E\ m(r, \frac{f'}{f}) = (1+\varepsilon)\log\gamma(r) + \varepsilon \log r + O(\log T(r,f)).$$

Then, for any small function a (with respect to f) we also have

$$\|_E\ m(r, \frac{a'}{a}) = (1+\varepsilon)\log\gamma(r) + \varepsilon \log r + o(T(r,f)).$$

This implies that

$$\|_E\ N(r,\nu^0_{\frac{a'}{a}}) \le T(r,\frac{a'}{a}) = N(r,\nu^\infty_{\frac{a'}{a}}) + m(r,\frac{a'}{a})$$
$$\le N^{[1]}(r,\nu^0_a) + N^{[1]}(r,\nu^\infty_a) + (1+\varepsilon)\log\gamma(r) + \varepsilon\log r + o(T(r,f)).$$

Remark.

- If f is of finite growth index (i.e., $c_f < +\infty$) then the Lemma 2.1, we may take $\gamma(r) = e^{(c_f+\varepsilon)T(r,f)}$.
- If $R = +\infty$, we may take $c_f = 0$.

Theorem 2.1 (First main theorem [7])**.** *Let f be a meromorphic function on $\Delta(R)$. Then for each $a \in \mathbb{C}$, we have*

$$T(r,f) = T(r,\frac{1}{f-a}) + o(T(r,f)).$$

The following theorem is due to S. D. Quang [8].

Theorem 2.2 (see [8, Theorem 1.1])**.** *Let f be a non-constant meromorphic function on $\Delta(R)$ and $a_1,...,a_5$ be five distinct small functions (with respect to f). Assume that $\gamma(r)$ be a non-negative measurable function defined on $(0,R)$ with $\int_0^R \gamma(r)dr = +\infty$. Then, for any $\varepsilon > 0$, it holds that*

$$\|_E\ 2T(r,f) \le \sum_{i=1}^{5} N^{[1]}(r,\nu^0_{f-a_i}) + 19((1+\varepsilon)\log\gamma(r) + \varepsilon\log r)) + o(T(r,f).$$

From Theorem 2.2, we easily get the following result.

Theorem 2.3 (Second main theorem)**.** *Let f be a non-constant meromorphic function on $\Delta(R)$ and $a_1,...,a_q$ be distinct small functions (with respect to f). Assume that $\gamma(r)$ is a non-negative measurable function defined on $(0,R)$ with $\int_0^R \gamma(r)dr = +\infty$. Then, for any $\varepsilon > 0$,*

$$\|_E\ \frac{2q}{5}T(r,f) \le \sum_{i=1}^{q} N^{[1]}(r,\nu^0_{f-a_i}) + 19((1+\varepsilon)\log\gamma(r) + \varepsilon\log r)) + o(T(r,f).$$

3. Proof of Theorems 1.1

In order to prove Theorem 1.1, we need the following auxiliary result.

Lemma 3.1. *Let f be a nonconstant meromorphic function on a disc $\Delta(R)$ and a be a small function (with respect to f). Then, for any $\varepsilon > 0$ and positive integer k (maybe $k = +\infty$), we have*

$$kN^{[1]}(r, \nu^0_{f-a,>k}) \leq N(r, \nu^0_{f-a}) - N^{[1]}(r, \nu^0_{f-a}).$$

Proof. We have

$$\begin{aligned} N^{[1]}(r, \nu^0_{f-a}) &= N^{[1]}(r, \nu^0_{f-a,\leq k}) + N^{[1]}(r, \nu^0_{f-a,>k}) \\ &\leq N^{[1]}(r, \nu^0_{f-a,\leq k}) + \frac{1}{k+1} N(r, \nu^0_{f-a,>k}) \\ &\leq \frac{k}{k+1} N^{[1]}(r, \nu^0_{f-a,\leq k}) + \frac{1}{k+1} N^{[1]}(r, \nu^0_{f-a,\leq k}) + \frac{1}{k+1} N(r, \nu^0_{f-a,>k}) \\ &\leq \frac{k}{k+1} N^{[1]}(r, \nu^0_{f-a,\leq k}) + \frac{1}{k+1} N(r, \nu^0_{f-a}). \end{aligned}$$

This implie that

$$(k+1)N^{[1]}(r, \nu^0_{f-a}) \leq kN^{[1]}(r, \nu^0_{f-a,\leq k}) + N(r, \nu^0_{f-a}).$$

Thus

$$kN^{[1]}(r, \nu^0_{f-a,>k}) \leq N(r, \nu^0_{f-a}) - N^{[1]}(r, \nu^0_{f-a}).$$

The lemma is proved. □

Lemma 3.2. *Let f and g be two distinct meromorphic functions on $\Delta(R)$ with finite growth indices c_f and c_g, respectively, and $a_1, ..., a_q (q \geq 5)$ be distinct small functions with respect to f and g. Suppose that*

$$\min\{1, \nu^0_{f-a_i,\leq k_i}\} = \min\{1, \nu^0_{g-b_i,\leq k_i}\} \ (1 \leq i \leq q).$$

Let ε be a positive real number. Setting $T(r) = T(r, f) + T(r, g), \gamma(r) = e^{(\varepsilon + \max\{c_f, c_g\})T(r)}$ and $S(r) = (1+\varepsilon)\log\gamma(r) + \varepsilon \log r$, then we have

$$\|_E \sum_{i=5}^{q} N^{[1]}(r, \nu^0_{f-a_i=g-a_i}) \leq N^{[1]}(r, \nu^0_{f-a_i,>k}) + N^{[1]}(r, \nu^0_{g-a_i,>k}) + 7S(r) + o(T(r)). \tag{3..1}$$

Here, $N^{[1]}(r, \nu^0_{f-a=g-a})$ denotes the counting function without multiplicity which counts all common zeros of $f - a$ and $g - a$, and $N^{[1]}(r, \nu_{f-a,>k})$ denotes the counting function without multiplicity which counts zero of $f - a$ with multiplicity at least $k+1$.

Proof. If $\sum_{i=5}^q N^{[1]}(r, N^0_{f-a_i=g-a_i}) = o(T(r))$ then (3..1) obviously holds. Now, we suppose $\sum_{i=5}^q \overline{N}^{[1]}(r, \nu^0_{f-a_i=g-a_i}) \neq o(T(r))$. We set $\mathcal{V} = \bigcup_{1\leq i<j\leq q} \sup(\nu^0_{a_i-a_j})$. Then

$\mathcal{V}$ is a discrete subset of $\Delta(R)$ and $N(r,\mathcal{V}) = o(T(r))$, where $N(r,\mathcal{V})$ is the counting function without multiplicity which counts all points in $\mathcal{V}$. By using the quasi-Möbius transformation

$$L(w) = \frac{(\omega - a_1)(a_3 - a_2)}{(\omega - a_2)(a_3 - a_1)}$$

and considering two functions $L(f), L(g)$ if necessary, we may assume that $a_1 = 0, a_2 = \infty, a_3 = 1$ and $a_4 = a$ with $a \notin \{0, \infty, 1\}$ (this quasi-Möbius transformation only make the counting functions in the inequality of the lemma change up to small terms $o(T(r))$. We denote by V_u $(u \in \{0, \infty, 1, a\})$ the set of points which are either zero of $f - u$ or zero of $g - u$, where $f - \infty$ is regarded as $\frac{1}{f}$.

Now we set

$$H = \frac{f'(a'g - ag')(f - g)}{f(f-1)g(g-a)} - \frac{g'(a'f - af')(f-g)}{g(g-1)f(f-a)}. \quad (3..2)$$

Then

$$H = \frac{(f-g)Q}{f(f-1)(f-a)g(g-1)(g-a)}, \quad (3..3)$$

where

$$\begin{aligned} Q &= f'(a'g - ag')(f-a)(g-1) - g'(a'f - af')(g-a)(f-1) \\ &= a'ff'g^2 - a'ff'g - a(a-1)ff'g' - aa'f'g^2 + aa'f'g \\ &- a'f^2gg' + a'fgg' + a(a-1)f'gg' + aa'f^2g' - aa'fg'. \end{aligned} \quad (3..4)$$

Case 1: Suppose that $H \equiv 0$. Then from (3..2), we have

$$\frac{f'(a'g - ag')}{(f-1)(g-a)} \equiv \frac{g'(a'f - af')}{(g-1)(f-a)}.$$

This implie that

$$\begin{aligned} \frac{(f-g)(1-a)}{(g-1)(f-a)} &= \frac{(f-1)(g-a)}{(g-1)(f-a)} - 1 = \frac{f'(a'g - ag')}{g'(a'f - af')} - 1 \\ &= \frac{a'[(f'-g')g - (f-g)g']}{g'(a'f - af')}. \end{aligned}$$

This yields that

$$\frac{f'-g'}{f-g} = \frac{(1-a)g'(a'f - af')}{a'g(g-1)(f-a)} + \frac{g'}{g}. \quad (3..5)$$

Hence, if there exists a point $z_0 \notin \mathcal{V}$ which is a common zero of $f - a_i$ and $g - a_i$ $(5 \leq i \leq q)$ then it must be a pole of the left hand side of (3..5) but not be pole of the right hand side. This is a contradiction. Thus,

$$\sum_{i=5}^{q} N^{[1]}(r, N^0_{f-a_i=g-a_i}) \leq (q-4)N^{[1]}(r,\mathcal{V}) = o(T(r)).$$

Case 2: Suppose that $H \not\equiv 0$. From (3..2) and (3..4), we easily see that if $z \notin \mathcal{V}$ is a common zero of $f - a_i$ and $g - a_i$ $(5 \leq i \leq q)$ then it is a zero of $f - g$ and is not a pole of $\frac{Q}{f(f-1)(f-a)g(g-1)(g-a)}$. Hence it is a zero of H. Therefore,

$$\begin{aligned}\sum_{i=5}^{q} N^{[1]}(r, \nu^0_{f-a_i=g-a_i}) &\leq N^{[1]}(r, \nu^0_H) + N(r, \mathcal{V}) + o(T(r))\\ &\leq T(r, H) + o(T(r))\\ &\leq m(r, H) + N(r, \nu^\infty_H) + o(T(r)).\end{aligned} \tag{3..6}$$

We now estimate the proximity function $m(r, H)$. First, we have

$$\begin{aligned}H =&\frac{f'}{f-1}\frac{a'g-ag'}{g(g-a)} - (\frac{f'}{f-1} - \frac{f'}{f})\frac{a'g-ag'}{g-a}\\ &- \frac{g'}{g-1}\frac{a'f-af'}{f(f-a)} - (\frac{g'}{g-1} - \frac{g'}{g})\frac{a'f-af'}{f-a}\\ =&\frac{f'}{f-1}(\frac{g'}{g} - \frac{g'-a'}{g-a}) - (\frac{f'}{f-1} - \frac{f'}{f})(a' - a\frac{g'-a'}{g-a})\\ &- \frac{g'}{g-1}(\frac{f'}{f} - \frac{f'-a'}{f-a}) - (\frac{g'}{g-1} - \frac{g'}{g})(a' - a\frac{f'-a'}{f-a}).\end{aligned} \tag{3..7}$$

By the lemma on logarithmic derivatives, we get

$$\begin{aligned}m(r, H) \leq& m\left(r, \frac{f'}{f}\right) + m\left(r, \frac{g'}{g}\right) + m\left(r, \frac{f'}{f-1}\right) + m\left(r, \frac{g'}{g-1}\right)\\ &+ m\left(r, \frac{f'-a'}{f-a}\right) + m\left(r, \frac{g'-a'}{g-a}\right) + m\left(r, \frac{a'}{a}\right)\\ &\leq 7S(r) + o(T(r)).\end{aligned} \tag{3..8}$$

We now estimate the counting function $N(r, \nu^\infty_H)$. From (3..7), we know that the poles of H only possibly occur from the zeros of $f - a_i, g - a_i$, $(i \in \{1, 2, 3, 4\})$. We consider the following four subcases.

Subcase 1: z is a pole of a' or a. Hence z must be a pole of a. We note that each pole of every meromorphic function of the form $\dfrac{h'}{h}$ has multiplicity at most 1. Therefore,

$$N(r, \nu^\infty_H) \leq N(r, \nu^\infty_a) + 2 \leq 3N(r, \nu^\infty_a) = o(T(r)).$$

Subcase 2: z is not a pole of a and z is a common zero of $(f - u)$ and $(g - u)$ for a function $u \in \{0, \infty, 1, a\}$. From (3..4), we rewrite H as follows:

$$\begin{aligned}H &= (f-g)\left[(\frac{f'}{f-1} - \frac{f'}{f})(\frac{g'}{g} - \frac{g'-a'}{g-a}) - (\frac{g'}{g-1} - \frac{g'}{g})(\frac{f'}{f} - \frac{f'-a'}{f-a})\right]\\ &= (f-g)P,\end{aligned}$$

where
$P = \left[\frac{f'}{f-1}\frac{g'}{g} - \frac{f'}{f-1}\frac{g'-a'}{g-a} + \frac{f'}{f}\frac{g'-a'}{g-a} - \frac{g'}{g-1}\frac{f'}{f} + \frac{g'}{g-1}\frac{f'-a'}{f-a} - \frac{f'-a'}{f-a}\frac{g'}{g}\right].$
Hence, z is a zero of $f-g$ and a simple pole of P. Therefore z is not a pole of H.

Subcase 3: z is not a pole of a and is a common pole of f and g. From (3..3) and (3..4), we easily see that z is not a pole of H.

Subcase 4: z is not a pole of a and z is either a zero of $f-a_i$ or a zero of $g-a_i$ for some $i \in \{1, ..., 4\}$. From (3..7), H has the following form

$$H = \sum_{\substack{u,v \in \{0,\infty,1,a\} \\ u \neq v}} a_{uv} \frac{f'-u' ='}{f-u}\frac{g'-v'}{g-v},$$

where a_{uv} are constants or $\pm a'$ or $\pm a$. Hence

$$\begin{aligned} N(r,\nu_H^\infty) &\leq \max_{\substack{u,v \in \{0,\infty,1,a\} \\ u \neq v}} (N(r,\nu^\infty_{\frac{f'-u'}{f-u}}) + N(r,\nu^\infty_{\frac{g'-v'}{g-v}})) \\ &\leq \sum_{i=1}^{4} \left(N^{[1]}(r,\nu^0_{f-a_i}) + N^{[1]}(r,\nu^0_{g-a_i}) - N^{[1]}(r,\nu^0_{f-a_i=g-a_i})\right). \end{aligned}$$

From the above four case, we have

$$\begin{aligned} N(r,\nu_H^\infty) &\leq \sum_{i=1}^{4} \left(N^{[1]}(r,\nu^0_{f-a_i}) + N^{[1]}(r,\nu^0_{g-a_i}) - N^{[1]}(r,\nu^0_{f-a_i=g-a_i})\right) + o(T(r)) \\ &\leq \sum_{i=1}^{4} \left(N^{[1]}(r,\nu^0_{f-a_i,>k_i}) + N^{[1]}(r,\nu^0_{g-a_i,>k_i})\right) + o(T(r)). \end{aligned}$$

Combining the above inequality and (3..6), (3..8), we get

$$\sum_{i=5}^{q} N^{[1]}(r,\nu^0_{f-a_i=g-a_i}) \leq \sum_{i=1}^{4} \left(N^{[1]}(r,\nu^0_{f-a_i,>k_i}) + N^{[1]}(r,\nu^0_{g-a_i,>k_i})\right) + 7S(r) + o(T(r)).$$

The lemma is proved in this case. □

Proof of Theorem 1.1.

Proof. By Lemma 3.2 for every subset $\{i_1, ..., i_4\}$, we have

$$\begin{aligned}
&\sum_{i=1}^{q}\big(N^{[1]}(r,\nu^0_{f-a_i})+N^{[1]}(r,\nu^0_{g-a_i})\big)-\sum_{j=1}^{4}\Big(N^{[1]}(r,\nu^0_{f-a_{i_j}})+N^{[1]}(r,\nu^0_{g-a_{i_j}})\Big)\\
&\quad=\sum_{j=5}^{q}\Big(N^{[1]}(r,\nu^0_{f-a_{i_j}})+N^{[1]}(r,\nu^0_{g-a_{i_j}})\Big)\\
&\quad\le\sum_{j=5}^{q}\Big(2N^{[1]}(r,\nu^0_{f-a_{i_j}=g-a_{i_j}})+N^{[1]}(r,\nu^0_{f-a_{i_j},>k_{i_j}})+N^{[1]}(r,\nu^0_{g-a_{i_j},>k_{i_j}})\Big)\\
&\quad\le 2\sum_{j=1}^{4}\Big(N^{[1]}(r,\nu^0_{f-a_{i_j},>k_{i_j}})+N^{[1]}(r,\nu^0_{g-a_{i_j},>k_{i_j}})\Big)\\
&\quad+\sum_{j=5}^{q}\Big(N^{[1]}(r,\nu^0_{f-a_{i_j},>k_{i_j}})+N^{[1]}(r,\nu^0_{g-a_{i_j},>k_{i_j}})\Big)+7T(r)+o(T(r))\\
&\quad\le\sum_{i=1}^{q}\big(N^{[1]}(r,\nu^0_{f-a_i,>k_i})+N^{[1]}(r,\nu^0_{g-a_i,>k_i})\big)\\
&\quad+\sum_{j=1}^{4}\Big(N^{[1]}(r,\nu^0_{f-a_{i_j},>k_{i_j}})+N^{[1]}(r,\nu^0_{g-a_{i_j},>k_{i_j}})\Big)+7T(r)+o(T(r)).
\end{aligned}$$

By summing-up both sides of the above inequality over all $1 \le i_1 < i_2 < i_3 < i_4 \le q$ and utilizing Lemma 3.1, we obtain

$$\begin{aligned}
&(q-4)\sum_{i=1}^{q}\big(N^{[1]}(r,\nu^0_{f-a_i})+N^{[1]}(r,\nu^0_{g-a_i})\big)\\
&\quad\le(q+4)\sum_{i=1}^{q}\big(N^{[1]}(r,\nu^0_{f-a_i,>k_i})+N^{[1]}(r,\nu^0_{g-a_i,>k_i})\big)+7qS(r)+o(T(r))\\
&\quad\le(q+4)\sum_{i=1}^{q}\frac{1}{k_i}\big(N(r,\nu^0_{f-a_i})-N^{[1]}(r,\nu^0_{f-a_i})+N(r,\nu^0_{g-a_i})-N^{[1]}(r,\nu^0_{g-a_i})\big)\\
&\quad+7qS(r)+o(T(r)).
\end{aligned}$$

Thus

$$\begin{aligned}
(q-4+\frac{q+4}{k_0})\sum_{i=1}^{q}\big(N^{[1]}(r,\nu^0_{f-a_i})+N^{[1]}(r,\nu^0_{g-a_i})\big)&\le\sum_{i=1}^{q}\frac{q+4}{k_i}\big(N(r,\nu^0_{f-a_i})+N(r,\nu^0_{g-a_i})\big)\\
&+7qS(r)+o(T(r))\\
&\le\sum_{i=1}^{q}\frac{q+4}{k_i}\left(T(r,f)+T(r,g)\right)\\
&+7qS(r)+o(T(r)).
\end{aligned}$$

From this and Theorem 2.3, we get

$$(q-4+\frac{q+4}{k_0})\frac{q}{5}(2T(r)-38S(r)) \le \sum_{i=1}^{q}\frac{q+4}{k_i}T(r)+7qS(r)+o(T(r)).$$

This yields that

$$\|_E \ \frac{qk_0(2q-8)+2q(q+4)-5(q+4)\sum_{i=1}^{q}\frac{1}{k_i}}{qk_0(38q-117)+38q(q+4)}T(r) \le S(r)+o(T(r)).$$

Let $\varepsilon \to 0$ and then $r \to R$ ($r \notin E$), we obtain

$$c_f + c_g \ge 2\min\{c_f, c_g\} \ge \frac{qk_0(2q-8)+2q(q+4)-5(q+4)\sum_{i=1}^{q}\frac{1}{k_i}}{qk_0(19q-\frac{117}{2})+19q(q+4)}.$$

This is a contradiction. The theorem is proved. □

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