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A NEW METHOD FOR REGULARIZING A HEAT INVERSE PROBLEM WITH TIME DEPENDENT COEFFICIENT IN THE TWO-DIMENSIONAL CASE

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Abstract. In this paper, we consider an inverse problem with a time-dependent coefficient. This problem is ill-posed. To regularize this problem, we use the integral truncation method combined with the quasi-boundary value method. We construct approximate solutions and consider the stability of such a solution. Moreover, we evaluate the errors between regularized solutions and exact solutions. A numerical method is given to illustrate the theoretically obtained results.
Keywords: heat inverse problem, regularization, truncation method, quasi-boundary value method.

1. Introduction

We consider the problem of finding the temperature distribution u satisfying

$$\begin{cases} u_t + A(t)u = 0, t \in [0,T] \\ u(T) = g \end{cases} \tag{1.1}$$

where $A(t)$ is a linear operator given in an appropriate function space. This problem has a strong background in physics and engineering and has been researched by many scientists. As we know, this problem is severely ill-posed, i.e., the solution does not always exist and when it exists, it does not depend on the data continuously. A small error in the data can cause a large error in the solution. So, a regularization method for this problem is necessary. It has been considered by many authors using different methods in previous papers (see [1]-[6]).

For example, in [1], Lattes - Lions and in [2], Clark - Oppenheimer used the quasi-reversibility method to regularize problem (1.1) by changing the main equation as follows:

$$\begin{cases} u_t + Au - \varepsilon A^* Au = 0, & t \in [0,T] \\ u(T) = g \end{cases} \tag{1.2}$$

Another method is the quasi-boundary value method, in which the authors adjust the boundary conditions. For instance, in [3], [4], Denche - Bessila considered the boundary conditions: $u(T) + \varepsilon u(0) = g$ or $u(T) - \varepsilon u'(0) = g$. In [5], [6], Quan et al. proposed a modified quasi-boundary value method for problem (1.1).

Besides that, a regularization method for problem (1.1) which is often used by many mathematicians for the inverse problem for heat equation is the integral truncation method. Using this method, they cut - off the high frequency term in the solution to get the approximate solution.

Motivated by these reasons, in this present paper, we consider problem (1.1) with $A(t)u = -\frac{1}{a(t)}[u_{xx} + u_{yy}]$ in the two - dimensional case as follows:

Finding the temperature distribution $u(x,y,t)$ satisfying

$$\begin{cases} u_{xx}(x,y,t) + u_{yy}(x,y,t) = a(t)u_t(x,y,t), & (x,y,t) \in \mathbb{R}^2 \times [0,T] \\ u(x,y,T) = g(x,y), \quad (x,y) \in \mathbb{R}^2 \end{cases} \tag{1.3}$$

where

- $g \in L^2(\mathbb{R}^2)$ is given.
- $a \in C[0,T]$ such that there exist $M, N > 0 : M \le \frac{1}{a(t)} \le N$, $\forall t \in [0,T]$.

To regularize problem (1.3), we used a new method, which is the association of the integral truncation method and the quasi - boundary value method. With different conditions on the exact solution, we will get the error estimates of Hölder type or logarithmic type between the regularized solution and the exact solution.

2. Auxiliary results

Lemma 2.1. *(see [5]) Let* $0 < \varepsilon < M$ *and* $g(k) = \frac{1}{\varepsilon k + e^{-kM}}$*, we have*

$$g(k) \le \frac{M}{\varepsilon\left(1 + \ln\left(\frac{M}{\varepsilon}\right)\right)} \le \frac{M}{\varepsilon \ln\left(\frac{M}{\varepsilon}\right)}, \quad \forall k \ge 0.$$

Lemma 2.2. *(see [5]) Let* $0 \le t \le s \le M, 0 < \varepsilon < M, \xi, \omega \in \mathbb{R}$ *and* $M_1 = \max\{1, M\}$*, we have*

- $\frac{e^{(s-t-M)\left(\xi^2+\omega^2\right)}}{\varepsilon(\xi^2+\omega^2)+e^{-M(\xi^2+\omega^2)}} \le M_1 \left[\varepsilon \ln\left(\frac{M}{\varepsilon}\right)\right]^{\frac{t-s}{M}}$,

- $\frac{e^{-t\left(\xi^2+\omega^2\right)}}{\varepsilon(\xi^2+\omega^2)+e^{-M(\xi^2+\omega^2)}} \le M_1\left[\varepsilon\ln\left(\frac{M}{\varepsilon}\right)\right]^{\frac{t-M}{M}}$.

Lemma 2.3. *(see [6]) Let $n \in \mathbb{R}, 0 \le a \le b, b \ne 0$, we have*

$$\frac{e^{na}}{1+\gamma e^{nb}} \le \gamma^{-\frac{a}{b}}, \forall \gamma > 0.$$

3. The solution and the ill-posedness of problem (1.1)

In the following sections, we denote $\|.\|_2$ the $L^2(\mathbb{R}^2)$-norm. Using the Fourier transform, we get the exact solution of problem (1.1) as follows:

$$u(x,y,t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} e^{\left(\xi^2+\omega^2\right)(F(T)-F(t))}\hat{g}(\xi,\omega)e^{i(\xi x+\omega y)}d\xi d\omega, \tag{3.1}$$

where

$$\hat{g}(\xi,\omega) = \frac{1}{2\pi}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} g(x,y)e^{-i(\xi x+\omega y)}dxdy,$$

and

$$F(t) = \int_0^t \frac{1}{a(s)}ds.$$

Next, we prove the ill-posedness of problem (1.3).

Choosing the exact data $g = 0$, we have the exact solution $u_{ex} = 0$.

Let us choose the measured data $\hat{g}_n = \begin{cases} 0, & \min\{\xi,\omega\} \le n, \\ \frac{n}{(\xi\omega)^{5/4}}, & \min\{\xi,\omega\} > n, \end{cases} \quad (n \in \mathbb{N})$.

Using Plancherel's theorem, we get the error of the data

$$\|g_n - g\|_2^2 = \|\hat{g}_n - \hat{g}\|_2^2 = \int_n^{+\infty}\int_n^{+\infty}\frac{n^2}{\xi^{5/2}\omega^{5/2}}d\xi d\omega = \frac{4}{9n} \to 0 \text{ as } n \to +\infty.$$

This implies that $g_n \to g$ as $n \to +\infty$.

We have the Fourier transform of the exact solution of problem (1.3) corresponding to the measured g_n as follows:

$$\hat{u}_n(g_n)(\xi,\omega,t) = \begin{cases} 0, & \min\{\xi,\omega\} \le n, \\ \frac{ne^{\left(\xi^2+\omega^2\right)(F(T)-F(t))}}{(\xi\omega)^{5/4}}, & \min\{\xi,\omega\} > n, \end{cases} \quad (n \in \mathbb{N}).$$

We get

$$\|\hat{u}_n(g_n)(.,.,t) - \hat{u}_{ex}(g)(.,.,t)\|_2^2 = \int_n^{+\infty}\int_n^{+\infty}\frac{n^2e^{2\left(\xi^2+\omega^2\right)(F(T)-F(t))}}{\xi^{5/2}\omega^{5/2}}d\xi d\omega.$$

Using $F(T)-F(t)=\int_t^T \frac{1}{a(s)}ds \geq M(T-t),\ \forall t\in[0,T]$ and Plancherel's theorem, we obtain

$$\begin{aligned}\|u_n(g_n)(.,.,t)-u_{ex}(g)(.,.,t)\|_2^2 &\geq \int_n^{+\infty}\int_n^{+\infty}\frac{n^2e^{2\left(n^2+n^2\right)M(T-t)}}{\xi^{5/2}\omega^{5/2}}d\xi d\omega\\ &= \frac{4e^{4n^2M(T-t)}}{9n}\to+\infty,\end{aligned}$$

as $n\to+\infty$.

Thus, the solution of problem (1.3) is unstable. It leads to the ill-posedness of problem (1.3). So, the regularization is in order. In the next sections, we will give two regularization methods for this problem.

4. Main results

4.1. Integral truncation method associated with the quasi-boundary value method

We construct the first regularized solution for problem (1.3) as follows:

$$v_\varepsilon(x,y,t)=\frac{1}{2\pi}\int_{-c_\varepsilon}^{c_\varepsilon}\int_{-c_\varepsilon}^{c_\varepsilon}\frac{e^{-\left(\xi^2+\omega^2\right)F(t)}}{\beta+e^{-(\xi^2+\omega^2)F(T)}}\hat{g}(\xi,\omega)e^{i(\xi x+\omega y)}d\xi d\omega, \tag{4.1}$$

where $\beta=\beta(\varepsilon), c_\varepsilon=c(\varepsilon)$ such that $\lim_{\varepsilon\to0^+}\beta(\varepsilon)=0$ and $\lim_{\varepsilon\to0^+}c(\varepsilon)=+\infty$.

4.1.1. Stability of the first regularized solution

Lemma 4.1. *Let $\beta\in(0,F(T)), g_1,g_2\in L^2\left(\mathbb{R}^2\right)$ and $v_\varepsilon(g_1), v_\varepsilon(g_2)$ are two solutions given by (4.1) corresponding to the final data g_1, g_2, respectively. Then we get*

$$\|v_\varepsilon(g_1)(.,.,t)-v_\varepsilon(g_2)(.,.,t)\|_2\leq\beta^{\frac{M(t-T)}{NT}}\|g_1-g_2\|_2\,.$$

Proof. From (4.1), we have the regularized solutions corresponding to the final data g_1, g_2

$$v_\varepsilon(g_1)(x,y,t)=\frac{1}{2\pi}\int_{-c_\varepsilon}^{c_\varepsilon}\int_{-c_\varepsilon}^{c_\varepsilon}\frac{e^{-\left(\xi^2+\omega^2\right)F(t)}}{\beta+e^{-(\xi^2+\omega^2)F(T)}}\hat{g}_1(\xi,\omega)e^{i(\xi x+\omega y)}d\xi d\omega,$$

and

$$v_\varepsilon(g_2)(x,y,t)=\frac{1}{2\pi}\int_{-c_\varepsilon}^{c_\varepsilon}\int_{-c_\varepsilon}^{c_\varepsilon}\frac{e^{-\left(\xi^2+\omega^2\right)F(t)}}{\beta+e^{-(\xi^2+\omega^2)F(T)}}\hat{g}_2(\xi,\omega)e^{i(\xi x+\omega y)}d\xi d\omega.$$

We get

$$\begin{aligned}\|\hat{v}_\varepsilon(g_1)(.,.,t)-\hat{v}_\varepsilon(g_2)(.,.,t)\|_2^2 &= \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}|\hat{v}_\varepsilon(g_1)(.,.,t)-\hat{v}_\varepsilon(g_2)(.,.,t)|^2\,d\xi d\omega\\ &= \int_{-c_\varepsilon}^{c_\varepsilon}\int_{-c_\varepsilon}^{c_\varepsilon}\left|\frac{e^{-(\xi^2+\omega^2)F(t)}}{\beta+e^{-(\xi^2+\omega^2)F(T)}}\left(\hat{g}_1(\xi,\omega)-\hat{g}_2(\xi,\omega)\right)\right|^2 d\xi d\omega\end{aligned}$$

It follows from Lemma 2.3 and $Mt \le F(t) = \int_0^t \frac{1}{a(s)}ds \le Nt$ that

$$\begin{aligned}\|\hat{v}_\varepsilon(g_1)(.,.,t)-\hat{v}_\varepsilon(g_2)(.,.,t)\|_2^2 &\le \beta^{\frac{2M(t-T)}{NT}}\int_{-c_\varepsilon}^{c_\varepsilon}\int_{-c_\varepsilon}^{c_\varepsilon}|(\hat{g}_1(\xi,\omega)-\hat{g}_2(\xi,\omega))|^2\,d\xi d\omega\\ &\le \beta^{\frac{2M(t-T)}{NT}}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}|(\hat{g}_1(\xi,\omega)-\hat{g}_2(\xi,\omega))|^2\,d\xi d\omega\\ &\le \beta^{\frac{2M(t-T)}{NT}}\|\hat{g}_1-\hat{g}_2\|_2^2.\end{aligned}$$

Applying Plancherel's theorem, we have

$$\|v_\varepsilon(g_1)(.,.,t)-v_\varepsilon(g_2)(.,.,t)\|_2 \le \beta^{\frac{M(t-T)}{NT}}\|g_1-g_2\|_2.$$

This completes the proof of Lemma 4.1. □

4.1.2. Error estimation between the exact solution and the first regularized solution

Theorem 4.1. *Let $u_{ex}(.,.,t) \in L^2(\mathbb{R}^2)\,\forall t \in [0,T)$, $\varepsilon \in (0;\sqrt{F(T)})$, $g_\varepsilon, g_{ex} \in L^2(\mathbb{R}^2)$ such that $\|g_\varepsilon - g_{ex}\|_2 \le \varepsilon$ and $v_\varepsilon(g_\varepsilon)$ is the regularized solution given by (4.1) corresponding to the measured data. Assume that*

$$\iint_{R^2}\left|\left(1+\xi^2+\omega^2\right)\hat{u}_{ex}(\xi,\omega,t)\right|^2 d\xi d\omega \le H, \forall t\in[0,T].$$

Then we have

$$\|v_\varepsilon(g_\varepsilon)(.,.,t)-u_{ex}(.,.,t)\|_2 \le B(t)\left(\varepsilon^{\frac{NT}{NT+2M(T-t)}}+\frac{1}{\ln\left(\frac{1}{\varepsilon}\right)}\right), \forall t\in[0,T].$$

Proof. Using the triangle inequality, we get

$$\begin{aligned}&\|\hat{v}_\varepsilon(g_\varepsilon)(.,.,t)-\hat{u}_{ex}(.,.,t)\|_2\\ &\le \|\hat{v}_\varepsilon(g_\varepsilon)(.,.,t)-\hat{v}_\varepsilon(g_{ex})(.,.,t)\|_2+\|\hat{v}_\varepsilon(g_{ex})(.,.,t)-\hat{u}_{ex}(.,.,t)\|_2.\end{aligned}\tag{4.2}$$

From Lemma 4.1, we get

$$\begin{aligned}\|\hat{v}_\varepsilon(g_\varepsilon)(.,.,t)-\hat{v}_\varepsilon(g_{ex})(.,.,t)\|_2 &= \|v_\varepsilon(g_\varepsilon)(.,.,t)-v_\varepsilon(g_{ex})(.,.,t)\|_2\\ &\le \beta^{\frac{M(t-T)}{NT}}\|g_\varepsilon-g_{ex}\|_2 \qquad (4.3)\\ &\le \beta^{\frac{M(t-T)}{NT}}\varepsilon.\end{aligned}$$

On the other hand, we have

$$\|\hat{v}_\varepsilon(g_{ex})(.,.,t)-\hat{u}_{ex}(.,.,t)\|_2^2=\iint_{[-c_\varepsilon,c_\varepsilon]^2}|\hat{v}_\varepsilon(g_{ex})(\xi,\omega,t)-\hat{u}_{ex}(\xi,\omega,t)|^2d\xi d\omega$$
$$+\iint_{\mathbb{R}^2\backslash[-c_\varepsilon,c_\varepsilon]^2}|\hat{u}_{ex}(\xi,\omega,t)|^2d\xi d\omega. \quad (4.4)$$

First, we evaluate the former term

$$\begin{aligned}
&\iint_{[-c_\varepsilon,c_\varepsilon]^2}|\hat{v}_\varepsilon(g_{ex})(\xi,\omega,t)-\hat{u}_{ex}(\xi,\omega,t)|^2d\xi d\omega\\
&=\iint_{[-c_\varepsilon,c_\varepsilon]^2}\left|\frac{1}{\beta e^{(\xi^2+\omega^2)F(T)}+1}\hat{u}_{ex}(\xi,\omega,t)-\hat{u}_{ex}(\xi,\omega,t)\right|^2d\xi d\omega\\
&=\iint_{[-c_\varepsilon,c_\varepsilon]^2}\left|\frac{-\beta e^{\left(\xi^2+\omega^2\right)F(T)}}{\beta e^{(\xi^2+\omega^2)F(T)}+1}\hat{u}_{ex}(\xi,\omega,t)\right|^2d\xi d\omega\\
&=\iint_{[-c_\varepsilon,c_\varepsilon]^2}\left|-\beta e^{\left(\xi^2+\omega^2\right)F(T)}\right|^2\left|\left(1+\xi^2+\omega^2\right)\hat{u}_{ex}(\xi,\omega,t)\right|^2d\xi d\omega\\
&\le\beta^2e^{4c_\varepsilon^2NT}\iint_{\mathbb{R}^2}\left|\left(1+\xi^2+\omega^2\right)\hat{u}_{ex}(\xi,\omega,t)\right|^2d\xi d\omega\\
&\le H\beta^2e^{4c_\varepsilon^2NT}. \quad (4.5)
\end{aligned}$$

Next, we evaluate the latter term

$$\iint_{\mathbb{R}^2\backslash[-c_\varepsilon,c_\varepsilon]^2}|\hat{u}_{ex}(\xi,\omega,t)|^2d\xi d\omega=\int_{\mathbb{R}^2\backslash[-c_\varepsilon,c_\varepsilon]^2}\frac{|(1+\xi^2+\omega^2)\,\hat{u}_{ex}(\xi,\omega,t)|^2}{(1+\xi^2+\omega^2)^2}d\xi d\omega \quad (4.6)$$
$$\le\frac{H}{4c_\varepsilon^4}.$$

From (4.2), (4.3), (4.4), (4.5) and (4.6), we have

$$\|\hat{v}_\varepsilon(g_\varepsilon)(.,.,t)-\hat{u}_{ex}(.,.,t)\|_2\le\beta^{\frac{M(t-T)}{NT}}\varepsilon+\sqrt{H}\beta e^{2c_\varepsilon^2NT}+\frac{\sqrt{H}}{2c_\varepsilon^2}.$$

Choosing $\beta=\varepsilon^{\frac{2NT}{NT+2M(T-t)}}$, $c_\varepsilon=\sqrt{\frac{1}{4NT}\ln\left(\frac{1}{\beta}\right)}$, we get

$$\|\hat{v}_\varepsilon(g_\varepsilon)(.,.,t)-\hat{u}_{ex}(.,.,t)\|_2\le B(t)\left(\varepsilon^{\frac{NT}{NT+2M(T-t)}}+\frac{1}{\ln\left(\frac{1}{\varepsilon}\right)}\right),$$

where $B(t)=\max\left\{1+\sqrt{H},\frac{\sqrt{H}\left[MNT+2M^2(T-t)\right]}{N}\right\}$.

Applying Plancherel's theorem, we obtain

$$\|v_\varepsilon(g_\varepsilon)(.,.,t) - u_{ex}(.,.,t)\|_2 \le B(t)\left(\varepsilon^{\frac{NT}{NT+2M(T-t)}} + \frac{1}{\ln\left(\frac{1}{\varepsilon}\right)}\right).$$

This completes the proof of Theorem 4.1. □

Remark 4.1. *In Theorem 4.1, the error estimation is of logarithmic type. This is because that the imposed condition on the exact solution is not strong enough. In the next theorem, we consider a stronger condition on the exact solution to get a better error estimate.*

Theorem 4.2. *Let* $T \ge \frac{1}{M}, \varepsilon \in \left(0, (F(T))^{\frac{1}{MT}}\right), u_{ex}(.,.,t) \in L^2(\mathbb{R}^2)\,\forall t \in [0,T), g_\varepsilon, g_{ex} \in L^2(\mathbb{R}^2)$ *such that* $\|g_\varepsilon - g_{ex}\|_2 \le \varepsilon$ *and* $v_\varepsilon(g_\varepsilon)$ *is the regularized solution given by (4.1) corresponding to the measured data. Suppose that*

$$\iint_{\mathbb{R}^2}\left|e^{\xi^2+\omega^2}\hat{u}_{ex}(\xi,\omega,t)\right|^2 d\xi d\omega \le K_1,\ \forall t \in [0,T].$$

Then we have

$$\|v_\varepsilon(g_\varepsilon)(.,.,t) - u_{ex}(.,.,t)\|_2 \le \left(1 + 2\sqrt{K_1}\right)\varepsilon^{\frac{M}{M^2(T-t)+N}},\ \forall t \in [0,T].$$

Proof. It follows from (4.5) that

$$\begin{aligned}
\iint_{[-c_\varepsilon,c_\varepsilon]^2} &|\ \hat{v}_\varepsilon(g_{ex})(\xi,\omega,t) - \hat{u}_{ex}(\xi,\omega,t)|^2 d\xi d\omega \\
&= \iint_{[-c_\varepsilon,c_\varepsilon]^2}\left|\frac{1}{\beta e^{(\xi^2+\omega^2)F(T)}+1}\hat{u}_{ex}(\xi,\omega,t) - \hat{u}_{ex}(\xi,\omega,t)\right|^2 d\xi d\omega \\
&= \iint_{[-c_\varepsilon,c_\varepsilon]^2}\left|\frac{-\beta e^{\left(\xi^2+\omega^2\right)F(T)}}{\beta e^{(\xi^2+\omega^2)F(T)}+1}\hat{u}_{ex}(\xi,\omega,t)\right| d\xi d\omega
\end{aligned}$$

$$\begin{aligned}
&= \iint_{[-c_\varepsilon,c_\varepsilon]^2}\left|\frac{-\beta e^{\left(\xi^2+\omega^2\right)F(T)}}{e^{\xi^2+\omega^2}}\right|^2\left|e^{\xi^2+\omega^2}\hat{u}_{ex}(\xi,\omega,t)\right|^2 d\xi d\omega \\
&\le \beta^2 e^{4c_\varepsilon^2(NT-1)}\iint_{\mathbb{R}^2}\left|e^{\xi^2+\omega^2}\hat{u}_{ex}(\xi,\omega,t)\right|^2 d\xi d\omega \\
&\le K_1\beta^2 e^{4c_\varepsilon^2(NT-1)}.
\end{aligned} \tag{4.7}$$

On the other hand, we have

$$\iint_{\mathbb{R}^2\setminus[-c_\varepsilon,c_\varepsilon]^2}|\hat{u}_{ex}(\xi,\omega,t)|^2 d\xi d\omega = \int_{\mathbb{R}^2\setminus[-c_\varepsilon,c_\varepsilon]^2}\frac{\left|e^{\xi^2+\omega^2}\hat{u}_{ex}(\xi,\omega,t)\right|^2}{e^{2(\xi^2+\omega^2)}}d\xi d\omega \le \frac{K_1}{e^{4c_\varepsilon^2}}. \tag{4.8}$$

From (4.2), (4.3), (4.4), (4.7) and (4.8), it results in

$$\|\hat{v}_\varepsilon(g_\varepsilon)(.,.,t) - \hat{u}_{ex}(.,.,t)\|_2 \le \beta^{\frac{M(t-T)}{NT}}\varepsilon + \sqrt{K_1}\beta e^{2c_\varepsilon^2(NT-1)} + \frac{\sqrt{K_1}}{e^{2c_\varepsilon^2}}.$$

Choosing $\beta = \varepsilon^{\frac{MNT}{M^2(T-t)+N}}$, $c_\varepsilon = \sqrt{\frac{1}{2NT}\ln\left(\frac{1}{\beta}\right)}$, we get

$$\|\hat{v}_\varepsilon(g_\varepsilon)(.,.,t) - \hat{u}_{ex}(.,.,t)\|_2 \le \varepsilon^{\frac{M}{M^2(T-t)+N}} + \sqrt{K_1}\varepsilon^{\frac{M}{M^2(T-t)+N}} + \sqrt{K_1}\varepsilon^{\frac{M}{M^2(T-t)+N}}.$$

Applying Plancherel's theorem, we obtain

$$\|v_\varepsilon(g_\varepsilon)(.,.,t) - u_{ex}(.,.,t)\|_2 \le \left(1 + 2\sqrt{K_1}\right)\varepsilon^{\frac{M}{M^2(T-t)+N}}.$$

This completes the proof of Theorem 4.2. □

Remark 4.2. *1. In Theorem 4.2, the error estimate is of Hölder type. This estimate is sharp and better than that formulated in Theorem 4.1. However, a more restrictive condition on the exact solution is imposed in comparison to the one derived in Theorem 4.1.*

2. If $e^{\xi^2+\omega^2}\hat{g}(\xi,\omega) \in L^2(\mathbb{R}^2)$ then the derived condition on the exact solution will be satisfied. So, this condition is acceptable.

In the next section, we will give another regularization method for the problem (III).

4.2. Integral truncation method associated with the modified quasi-boundary value method

We construct the second regularized solution for problem (III)

$$w_\varepsilon(x,y,t) = \frac{1}{2\pi}\int_{-m_\varepsilon}^{m_\varepsilon}\int_{-m_\varepsilon}^{m_\varepsilon} \frac{e^{-\left(\xi^2+\omega^2\right)F(t)}}{\alpha\left(\xi^2+\omega^2\right) + e^{-(\xi^2+\omega^2)F(T)}}\hat{g}(\xi,\omega)e^{i(\xi x+\omega y)}d\xi d\omega, \quad (4.9)$$

where $\alpha = \alpha(\varepsilon), m_\varepsilon = m(\varepsilon)$ such that $\lim_{\varepsilon\to 0^+}\alpha(\varepsilon) = 0$ and $\lim_{\varepsilon\to 0^+} m(\varepsilon) = +\infty$.

4.2.1. Stability of the second regularized solution

Lemma 4.2. *Let $\varepsilon \in (0, F(T)), g_1, g_2 \in L^2(\mathbb{R}^2)$ and $w_\varepsilon(g_1), w_\varepsilon(g_2)$ are two solutions given by (3.9) corresponding to the final data g_1, g_2, respectively. Then we get*

$$\|w_\varepsilon(g_1)(.,.,t) - w_\varepsilon(g_2)(.,.,t)\|_2 \le \left[\alpha\ln\left(\frac{MT}{\alpha}\right)\right]^{\frac{N(t-T)}{MT}}\|g_1 - g_2\|_2.$$

Proof. From (4.9), we have the regularized solutions corresponding to the data g_1, g_2

$$w_\varepsilon(g_1)(x,y,t) = \frac{1}{2\pi}\int_{-m_\varepsilon}^{m_\varepsilon}\int_{-m_\varepsilon}^{m_\varepsilon} \frac{e^{-\left(\xi^2+\omega^2\right)F(t)}}{\alpha\left(\xi^2+\omega^2\right) + e^{-(\xi^2+\omega^2)F(T)}}\hat{g}_1(\xi,\omega)e^{i(\xi x+\omega y)}d\xi d\omega,$$

and

$$w_\varepsilon(g_2)(x,y,t) = \frac{1}{2\pi}\int_{-m_\varepsilon}^{m_\varepsilon}\int_{-m_\varepsilon}^{m_\varepsilon} \frac{e^{-\left(\xi^2+\omega^2\right)F(t)}}{\alpha\left(\xi^2+\omega^2\right)+e^{-\left(\xi^2+\omega^2\right)F(T)}}\hat{g}_2(\xi,\omega)e^{i(\xi x+\omega y)}d\xi d\omega.$$

We have

$$\begin{aligned}
&\left\|\hat{w}_\varepsilon(g_1)(.,.,t)-\hat{w}_\varepsilon(g_2)(.,.,t)\right\|_2^2 \\
&= \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\left|\hat{w}_\varepsilon(g_1)(.,.,t)-\hat{w}_\varepsilon(g_2)(.,.,t)\right|^2 d\xi d\omega \\
&= \int_{-m_\varepsilon}^{m_\varepsilon}\int_{-m_\varepsilon}^{m_\varepsilon}\left|\frac{e^{-\left(\xi^2+\omega^2\right)F(t)}}{\alpha\left(\xi^2+\omega^2\right)+e^{-\left(\xi^2+\omega^2\right)F(T)}}\left(\hat{g}_1(\xi,\omega)-\hat{g}_2(\xi,\omega)\right)\right|^2 d\xi d\omega
\end{aligned}$$

By using Lemma 2.2, we have

$$\begin{aligned}
&\left\|\hat{w}_\varepsilon(g_1)(.,.,t)-\hat{w}_\varepsilon(g_2)(.,.,t)\right\|_2^2 \\
&\le \left[\alpha\ln\left(\frac{F(T)}{\alpha}\right)\right]^{\frac{F(t)-F(T)}{F(T)}}\int_{-m_\varepsilon}^{m_\varepsilon}\int_{-m_\varepsilon}^{m_\varepsilon}(\hat{g}_1(\xi,\omega)-\hat{g}_2(\xi,\omega))^2 d\xi d\omega \\
&\le \left[\alpha\ln\left(\frac{F(T)}{\alpha}\right)\right]^{\frac{F(t)-F(T)}{F(T)}}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}(\hat{g}_1(\xi,\omega)-\hat{g}_2(\xi,\omega))^2 d\xi d\omega \\
&\le \left[\alpha\ln\left(\frac{F(T)}{\alpha}\right)\right]^{\frac{F(t)-F(T)}{F(T)}}\left\|\hat{g}_1-\hat{g}_2\right\|_2^2.
\end{aligned}$$

Applying Plancherel's theorem, we obtain

$$\left\|w_\varepsilon(g_1)(.,.,t)-w_\varepsilon(g_2)(.,.,t)\right\|_2 \le \left[\alpha\ln\left(\frac{F(T)}{\alpha}\right)\right]^{\frac{F(t)-F(T)}{F(T)}}\left\|g_1-g_2\right\|_2.$$

From $Mt \le F(t) = \int_0^t \frac{1}{a(s)}ds \le Nt$, we get

$$\left\|w_\varepsilon(g_1)(.,.,t)-w_\varepsilon(g_2)(.,.,t)\right\|_2 \le \left[\alpha\ln\left(\frac{MT}{\alpha}\right)\right]^{\frac{N(t-T)}{MT}}\left\|g_1-g_2\right\|_2.$$

This completes the proof of Lemma 4.2. □

4.2.2. Error estimation between the exact solution and the second regularized solution

Theorem 4.3. *Let* $T \geq \frac{1}{M}, \varepsilon \in (0, \sqrt{F(T)}), u_{ex}(.,.,t) \in L^2(\mathbb{R}^2)\, \forall t \in [0,T), g_\varepsilon, g_{ex} \in L^2(\mathbb{R}^2)$ *such that* $\|g_\varepsilon - g_{ex}\|_2 \leq \varepsilon$ *and* $w_\varepsilon(g_\varepsilon)$ *is the regularized solution corresponding to the measured data given by (4.9). Assume that*

$$\iint_{R^2} \left|\left(1+\xi^2+\omega^2\right)\hat{u}_{ex}(\xi,\omega,t)\right|^2 d\xi d\omega \leq H,\ \forall t \in [0,T].$$

Then we have

$$\begin{aligned}
&\|w_\varepsilon(g_\varepsilon)(.,.,t) - u_{ex}(.,.,t)\|_2 \\
&\leq L(t)\left[\varepsilon^{\frac{MT}{MT+2N(T-t)}}\left(\ln\left(\frac{1}{\varepsilon}\right)\right)^{\frac{N(t-T)}{MT}} + \varepsilon^{\frac{MT}{MT+2N(T-t)}}\ln\left(\frac{1}{\varepsilon}\right) + \left(\ln\left(\frac{1}{\varepsilon}\right)\right)^{-1}\right],
\end{aligned}$$

$\forall t \in [0,T]$.

Proof. Using Plancherel's theorem and triangle inequality, we obtain

$$\begin{aligned}
&\|w_\varepsilon(g_\varepsilon)(.,.,t) - u_{ex}(.,.,t)\|_2 \\
&= \|\hat{w}_\varepsilon(g_\varepsilon)(.,.,t) - \hat{u}_{ex}(.,.,t)\|_2 \\
&\leq \|\hat{w}_\varepsilon(g_\varepsilon)(.,.,t) - \hat{w}_\varepsilon(g_{ex})(.,.,t)\|_2 + \|\hat{w}_\varepsilon(g_{ex})(.,.,t) - \hat{u}_{ex}(.,.,t)\|_2 . \qquad (4.10)
\end{aligned}$$

From Lemma 4.2, we get

$$\begin{aligned}
\|\hat{w}_\varepsilon(g_\varepsilon)(.,.,t) - \hat{w}_\varepsilon(g_{ex})(.,.,t)\|_2 &= \|w_\varepsilon(g_\varepsilon)(.,.,t) - w_\varepsilon(g_{ex})(.,.,t)\|_2 \\
&\leq \left(\alpha\ln\left(\frac{MT}{\alpha}\right)\right)^{\frac{N(t-T)}{MT}}\|g_\varepsilon - g_{ex}\|_2 \\
&\leq \left(\alpha\ln\left(\frac{MT}{\alpha}\right)\right)^{\frac{N(t-T)}{MT}}\varepsilon. \qquad (4.11)
\end{aligned}$$

Moreover, we have

$$\begin{aligned}
\|\hat{w}_\varepsilon(g_{ex})(.,.,t) - \hat{u}_{ex}(.,.,t)\|_2^2 &= \iint_{[-m_\varepsilon,m_\varepsilon]^2}\left|\hat{w}_\varepsilon(g_{ex})(\xi,\omega,t) - \hat{u}_{ex}(\xi,\omega,t)\right|^2 d\xi d\omega \\
&+ \iint_{\mathbb{R}^2\setminus[-m_\varepsilon,m_\varepsilon]^2}\left|\hat{u}_{ex}(\xi,\omega,t)\right|^2 d\xi d\omega. \qquad (4.12)
\end{aligned}$$

First, we estimate the first integral

$$\iint_{[-m_\varepsilon,m_\varepsilon]^2} \left|\hat{w}_\varepsilon\left(g_{ex}\right)(\xi,\omega,t) - \hat{u}_{ex}(\xi,\omega,t)\right|^2 d\xi d\omega$$

$$= \iint_{[-m_\varepsilon,m_\varepsilon]^2} \left|\frac{1}{\alpha\left(\xi^2+\omega^2\right)e^{(\xi^2+\omega^2)F(T)}+1}\hat{u}_{ex}(\xi,\omega,t) - \hat{u}_{ex}(\xi,\omega,t)\right|^2 d\xi d\omega$$

$$= \iint_{[-m_\varepsilon,m_\varepsilon]^2} \left|\frac{-\alpha\left(\xi^2+\omega^2\right)e^{\left(\xi^2+\omega^2\right)F(T)}}{\alpha\left(\xi^2+\omega^2\right)e^{(\xi^2+\omega^2)F(T)}+1}\hat{u}_{ex}(\xi,\omega,t)\right|^2 d\xi d\omega$$

$$\le \iint_{[-m_\varepsilon,m_\varepsilon]^2} \left|-\alpha\left(\xi^2+\omega^2\right)e^{\left(\xi^2+\omega^2\right)F(T)}\right|^2 \left|\left(1+\xi^2+\omega^2\right)\hat{u}_{ex}(\xi,\omega,t)\right|^2 d\xi d\omega$$

$$\le 4\alpha^2 m_\varepsilon^4 e^{2m_\varepsilon^2 NT} \iint_{R^2} \left|\left(1+\xi^2+\omega^2\right)\hat{u}_{ex}(\xi,\omega,t)\right|^2 d\xi d\omega$$

$$\le 4H\alpha^2 m_\varepsilon^4 e^{2m_\varepsilon^2 NT}. \tag{4.13}$$

On the other hand, we have the second integral

$$\iint_{\mathbb{R}^2\setminus[-m_\varepsilon,d_\varepsilon]^2} \left|\hat{u}_{ex}(\xi,\omega,t)\right|^2 d\xi d\omega$$

$$= \int_{\mathbb{R}^2\setminus[-m_\varepsilon,m_\varepsilon]^2} \frac{\left|\left(1+\xi^2+\omega^2\right)\hat{u}_{ex}(\xi,\omega,t)\right|^2}{\left(1+\xi^2+\omega^2\right)^2} d\xi d\omega \le \frac{H}{4m_\varepsilon^4}. \tag{4.14}$$

From (4.10), (4.11), (4.12), (4.13) and (4.14), we obtain

$$\left\|\hat{w}_\varepsilon\left(g_\varepsilon\right)(.,.,t) - \hat{u}_{ex}(.,.,t)\right\|_2 \le \left[\alpha \ln\left(\frac{MT}{\alpha}\right)\right]^{\frac{N(t-T)}{MT}} \varepsilon + 2\sqrt{H}\alpha m_\varepsilon^2 e^{m_\varepsilon^2 NT} + \frac{\sqrt{H}}{2m_\varepsilon^2}.$$

Choosing $\alpha = \varepsilon^{\frac{2MT}{MT+2N(T-t)}}$, $m_\varepsilon = \sqrt{\frac{1}{2NT}\ln\left(\frac{1}{\alpha}\right)}$, we get

$$\left\|\hat{w}_\varepsilon\left(g_\varepsilon\right)(.,.,t) - \hat{u}_{ex}(.,,t)\right\|_2$$

$$\le \left(\frac{2MT}{MT+2N(T-t)}\right)^{\frac{N(t-T)}{MT}} \varepsilon^{\frac{MT}{MT+2N(T-t)}} \left[\ln\left(\frac{1}{\varepsilon}\right)\right]^{\frac{N(t-T)}{MT}}$$

$$+ \frac{2M\sqrt{H}}{NMT+2N^2(T-t)}\varepsilon^{\frac{MT}{MT+2N(T-t)}} \ln\left(\frac{1}{\varepsilon}\right) + \frac{\sqrt{H}(NMT+2N^2(T-t))}{2M\ln\left(\frac{1}{\varepsilon}\right)}.$$

It leads to an error estimate

$$\left\|w_\varepsilon\left(g_\varepsilon\right)(.,.,t) - u_{ex}(.,.,t)\right\|_2$$

$$\le L(t)\left[\varepsilon^{\frac{MT}{MT+2N(T-t)}}\left(\ln\left(\frac{1}{\varepsilon}\right)\right)^{\frac{N(t-T)}{MT}} + \varepsilon^{\frac{MT}{MT+2N(T-t)}}\ln\left(\frac{1}{\varepsilon}\right) + \left(\ln\left(\frac{1}{\varepsilon}\right)\right)^{-1}\right],$$

where $L(t) = \max\left\{ \left(\frac{2MT}{MT+2N(T-t)}\right)^{\frac{N(t-T)}{MT}}, \frac{2M\sqrt{H}}{NMT+2N^2(T-t)}, \frac{\sqrt{H}(NMT+2N^2(T-t))}{2M} \right\}$.

This completes the proof of Theorem 4.3. □

In the next section, to prove the effectiveness of our theoretical method, we give a numerical example.

5. Numerical example

We consider the problem of finding $u(x, y, t)$ that satisfies

$$u_{xx}(x, y, t) + u_{yy}(x, y, t) = a(t)u_t(x, y, t), \quad (x, y, t) \in \mathbb{R}^2 \times [0, 1], \tag{5.1}$$

and

$$u(x, y, 1) = g_{ex}(x, y), (x, y) \in \mathbb{R}^2, \tag{5.2}$$

where

$$a(t) = \frac{1}{4t + 3},$$

and

$$g_{ex}(x, y) = \frac{1}{16} e^{-\frac{x^2+y^2}{32}}.$$

The exact solution of $(5.1) - (5.2)$ is

$$u_{ex}(x, y, t) = \frac{1}{2\left(2t^2 + 3t + 3\right)} e^{-\frac{x^2+y^2}{4(2t^2+3t+3)}}$$

From (5.2), we have

$$\hat{u}_{ex}(\xi, \omega, 1) = \hat{g}_{ex}(\xi, \omega) = e^{-8\left(\xi^2+\omega^2\right)}. \tag{5.3}$$

From (5.3), we have the Fourier transform of the exact solution

$$\hat{u}_{ex}(\xi, \omega, t) = e^{(F(1)-F(t))\left(\xi^2+\omega^2\right)} \cdot e^{-8\left(\xi^2+\omega^2\right)}, \tag{5.4}$$

where

$$F(t) = \int_0^t \frac{1}{a(s)} ds = \int_0^t (4s + 3) ds = 2t^2 + 3t.$$

From (5.4), we have the Fourier transform of the exact solution at $t = 0$

$$\hat{u}_{ex}(\xi, \omega, 0) = e^{(F(1)-F(0))\left(\xi^2+\omega^2\right)} e^{-8\left(\xi^2+\omega^2\right)} = e^{-3\left(\xi^2+\omega^2\right)}. \tag{5.5}$$

Let us consider the perturbed data

$$g_\varepsilon(x, y) = \left(1 + \varepsilon\sqrt{\frac{16}{\pi}}\right) g_{ex}(x, y) = \left(1 + \varepsilon\sqrt{\frac{16}{\pi}}\right) \frac{1}{16} e^{-\frac{x^2+y^2}{32}}. \tag{5.6}$$

We have an error in the data

$$\|g_\varepsilon - g_{ex}\|_2 = \|\hat{g}_\varepsilon - \hat{g}_{ex}\|_2 = \varepsilon\sqrt{\frac{16}{\pi}}\left(\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\left|\frac{1}{16}e^{-\frac{x^2+y^2}{32}}\right|^2 dxdy\right)^{\frac{1}{2}} = \varepsilon.$$

Choosing $M = 3, N = 7$, we get

$$b_\varepsilon = \frac{1}{\sqrt{2N(T-t)+1}}\sqrt{\ln\left(\frac{1}{\varepsilon}\right)} = \frac{1}{\sqrt{15}}\sqrt{\ln\left(\frac{1}{\varepsilon}\right)},$$
$$\beta = \varepsilon^{\frac{MNT}{M^2(T-t)+N}} = \varepsilon^{\frac{21}{16}}, c_\varepsilon = \sqrt{\frac{1}{2NT}\ln\left(\frac{1}{\beta}\right)} = \sqrt{\frac{3}{32}\ln\left(\frac{1}{\varepsilon}\right)},$$
$$\alpha = \varepsilon^{\frac{2MT}{MT+2N(T-t)}} = \varepsilon^{\frac{6}{17}}, m_\varepsilon = \sqrt{\frac{1}{2NT}\ln\left(\frac{1}{\alpha}\right)} = \sqrt{\frac{3}{119}\ln\left(\frac{1}{\varepsilon}\right)}.$$

From (4.1), we have the Fourier transform of the first regularized solution at $t = 0$

$$\hat{v}_\varepsilon\left(g_\varepsilon\right)(\xi,\omega,0) = \frac{e^{-\left(\xi^2+\omega^2\right)F(0)}}{\beta + e^{-(\xi^2+\omega^2)F(1)}}\hat{g}_\varepsilon(\xi,\omega)\chi_{\left([-c_\varepsilon,c_\varepsilon]^2\right)}(\xi,\omega)$$
$$= \left(1+\varepsilon\sqrt{\frac{16}{\pi}}\right)\frac{e^{-8\left(\xi^2+\omega^2\right)}}{\varepsilon^{\frac{21}{16}} + e^{-5(\xi^2+\omega^2)}}\chi_{\left(\left[-\sqrt{\frac{3}{32}\ln\left(\frac{1}{\varepsilon}\right)},\sqrt{\frac{3}{32}\ln\left(\frac{1}{\varepsilon}\right)}\right]^2\right).}(\xi,\omega).$$

The error estimate between the first regularized solution $v_\varepsilon\left(g_\varepsilon\right)(.,.,0)$ and the exact solution $u_{ex}(.,,,0)$ is

$$\|v_\varepsilon\left(g_\varepsilon\right)(.,.,0) - u_{ex}(.,.,0)\|^2 = \|\hat{v}_\varepsilon\left(g_\varepsilon\right)(.,.,0) - \hat{u}_{ex}(.,.,0)\|^2$$
$$= \int_{-c_\varepsilon}^{c_\varepsilon}\int_{-c_\varepsilon}^{c_\varepsilon}\left[\left(1+\varepsilon\sqrt{\frac{16}{\pi}}\right)\frac{e^{-8\left(\xi^2+\omega^2\right)}}{\varepsilon^{\frac{21}{16}} + e^{-5(\xi^2+\omega^2)}} - e^{-3\left(\xi^2+\omega^2\right)}\right]^2 d\xi d\omega$$
$$+ \iint_{\mathbb{R}^2\backslash[-c_\varepsilon,c_\varepsilon]^2} e^{-6\left(\xi^2+\omega^2\right)}d\xi d\omega.$$

From (4.9), we get the Fourier transform of the second regularized solution at $t = 0$

$$\hat{w}_\varepsilon\left(g_\varepsilon\right)(\xi,\omega,0) = \frac{e^{-\left(\xi^2+\omega^2\right)F(0)}}{\alpha\left(\xi^2+\omega^2\right) + e^{-(\xi^2+\omega^2)F(1)}}\hat{g}_\varepsilon(\xi,\omega)\chi_{\left([-m_\varepsilon,m_\varepsilon]^2\right)}(\xi,\omega)$$
$$= \left(1+\varepsilon\sqrt{\frac{16}{\pi}}\right)\frac{e^{-8\left(\xi^2+\omega^2\right)}}{\varepsilon^{\frac{6}{17}}\left(\xi^2+\omega^2\right) + e^{-5(\xi^2+\omega^2)}}\chi_{\left(\left[-\sqrt{\frac{3}{119}\ln\left(\frac{1}{\varepsilon}\right)},\sqrt{\frac{3}{119}\ln\left(\frac{1}{\varepsilon}\right)}\right]^2\right)}(\xi,\omega).$$

The error estimate between the second regularized solution $w_\varepsilon(g_\varepsilon)(.,.,0)$ and the exact solution $u_{ex}(.,.,0)$ is

$$\|w_\varepsilon(g_\varepsilon)(.,.,0)-u_{ex}(.,.,0)\|^2=\|\hat{w}_\varepsilon(g_\varepsilon)(.,.,0)-\hat{u}_{ex}(.,.,0)\|^2$$

$$=\int_{-m_\varepsilon}^{m_\varepsilon}\int_{-m_\varepsilon}^{m_\varepsilon}\left[\left(1+\varepsilon\sqrt{\frac{16}{\pi}}\right)\frac{e^{-8\left(\xi^2+\omega^2\right)}}{\varepsilon^{\frac{6}{17}}\left(\xi^2+\omega^2\right)+e^{-5(\xi^2+\omega^2)}}-e^{-3\left(\xi^2+\omega^2\right)}\right]^2 d\xi d\omega$$

$$+\iint_{\mathbb{R}^2\setminus[-m_\varepsilon,m_\varepsilon]^2}e^{-6\left(\xi^2+\omega^2\right)}d\xi d\omega.$$

Table 1. The error estimate between the first regularized solution $v_\varepsilon(g_\varepsilon)(.,.,0)$ ***and the exact solution*** $u_{ex}(.,.,0)$

ε	$\|v_\varepsilon(g_\varepsilon)(.,.,0)-u_{ex}(.,.,0)\|_2$
10^{-1}	0.6304
10^{-5}	0.02262
10^{-10}	4.28×10^{-4}
10^{-15}	7.49×10^{-11}
10^{-30}	1.2068×10^{-16}

Table 2. The error estimate between the second regularized solution $w_\varepsilon(g_\varepsilon)(.,.,0)$ ***and the exact solution*** $u_{ex}(.,.,0)$

ε	$\|w_\varepsilon(g_\varepsilon)(.,.,0)-u_{ex}(.,.,0)\|_2$
10^{-1}	0.933
10^{-5}	0.156
10^{-10}	0.025
10^{-15}	2.168×10^{-3}
10^{-30}	3.9785×10^{-6}

Remark 5.1. *From Table 1 and Table 2, we can see that the error of the regularized solution and the exact solution is smaller when the error of the data is smaller. So, this computated result is consistent to the theoretical result, which demonstrates the effectiveness of our method.*

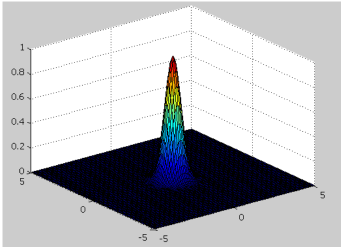

Figure 1. The graph of the Fourier transform of the exact solution $u_{ex}(.,.,0)$

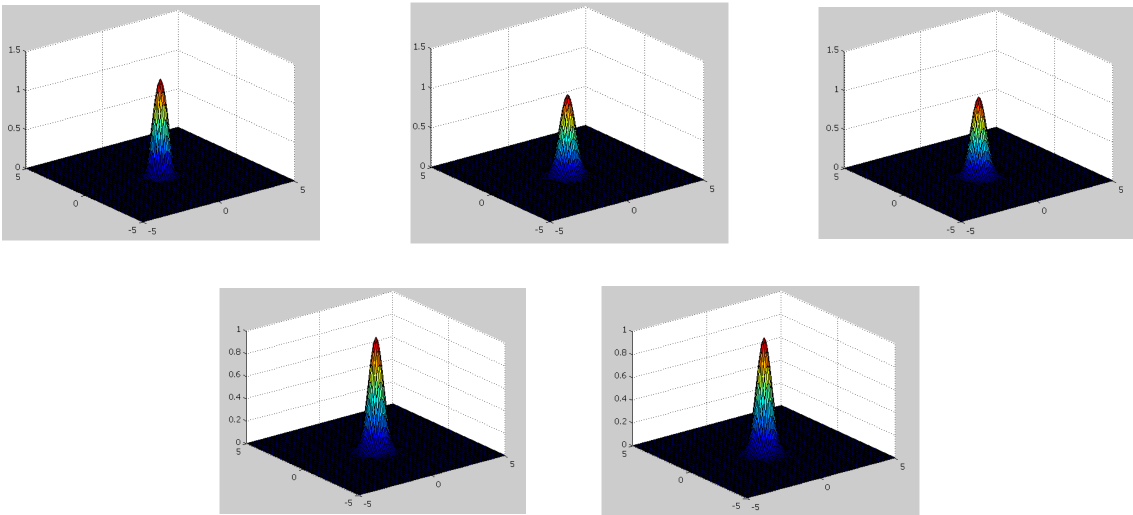

Figure 2. The graphs of the Fourier transform of the first regularized solutions $v_{\varepsilon_k}(g_{\varepsilon_k})(.,.,0), k = 1,2,3,4,5$

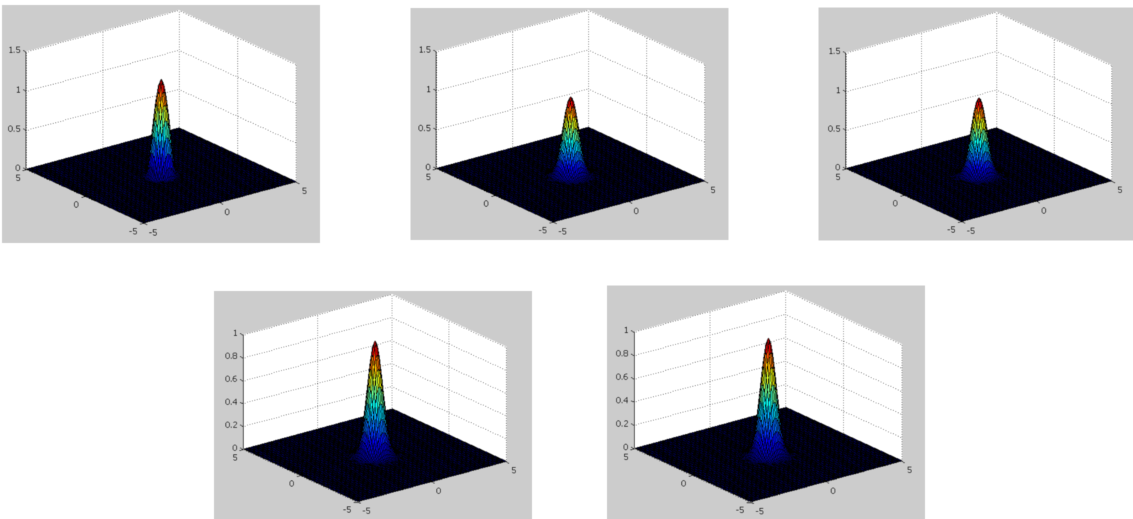

Figure 3. The graphs of the Fourier transform of the second regularized solutions $w_{\varepsilon_k}(g_{\varepsilon_k})(.,.,0), k = 1,2,3,4,5$

6. Conclusions

In this paper, we investigate the ill-posedness and regularization of a time-inverse heat problem with time-dependent coefficients in two-dimensional and homogeneous cases. In particular, we used the integral truncation method associated with quasi - boundary value method and modified the quasi-boundary value method to establish approximate solutions to the problem. Moreover, we established error estimates between the exact solution and the regularized solutions with different conditions on the exact solution. Finally, we provide a numerical example to illustrate the results obtained by our theoretical method. In the future, we will consider the problem in the nonhomogeneous case by using many methods.

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