

GREATEST COMMON DIVISOR OF SHIFTED PELL NUMBERS AND BALANCING NUMBERS

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Abstract. Let D_n denote the n 'th Pell, Pell -Lucas, Balancing or Lucas – Balancing number and let k be a positive integer. We find necessary and sufficient conditions on s and t so that the function $n \mapsto \gcd(D_n + s, D_{n+k} + t)$ is unbounded. Then, we give some nice applications for Diophantine equations.

Keywords: Pell, Pell-Lucas, balancing or Lucas - balancing numbers.

1. Introduction

A well-known recursive sequence is the Fibonacci sequence $(F_n)_{n \geq 0}$. It is defined by

$$F_0 = 0, F_1 = 1; F_n = F_{n-1} + F_{n-2}, n \geq 2.$$

It is easy to see that F_n and F_{n+1} are coprime for all integers $n \geq 0$. If we shift the values or indices, the situation changes, for example, $F_{4n+1} - 1$ and $F_{4n+2} - 1$ are divisible by F_{2n} . In 1971, Dudley and Tucker [1] showed that $\gcd(F_n + s, F_{n+1} + s)$ is unbounded for $s = \pm 1$. In 2011, Chen [2] determined $\gcd(F_n + s, F_{n+1} + s)$ for $s \in \{\pm 1, \pm 2\}$ and proved that $\gcd(F_n + s, F_{n+1} + s)$ is bounded for $s \neq \pm 1$. In 2018, Rahn and Kreh [3] determined $\gcd(F_n + s, F_{n+1} + s)$ for $s = \pm 3$. In 2016, Spilker [4] proved that $\gcd(F_n + s, F_{n+1} + s)$ divides $s^2 + (-1)^n$. In 2022, Hieu, Spilker, and Thang [5] investigated the necessary and sufficient conditions on $k \in \mathbb{N}, s, t \in \mathbb{Z}$, so that the functions $\gcd(F_n + s, F_{n+k} + s)$ and $\gcd(F_n + s, F_{n+1} + t)$ are unbounded and later, in 2022, Do and Thang [6] investigated the necessary and sufficient conditions on $k \in \mathbb{N}, s, t \in \mathbb{Z}$, so that the function $\gcd(F_n + s, F_{n+k} + t)$ is unbounded. Similar results on Lucas numbers are mentioned in [5], [6]. In this paper, we determine such exceptional shifts, not only for Fibonacci sequence (F_n) , Lucas sequences (L_n) , but also for the Pell sequence (P_n) , the associated Pell sequence (Q_n) , the balancing sequence (B_n) and the Lucas-balancing sequence (C_n) . Finally, we give some nice applications for Diophantine equations.

In this article, $\mathbb{N}$ denotes the set of positive integers, $\mathbb{N}_0$ denotes the set of non-negative integers, and $\mathbb{Z}$ denotes the set of integers.

2. Pell and Pell-Lucas sequences

The Pell sequence is defined by the recursion

$$P_0 = 0, P_1 = 1, P_n = 2P_{n-1} + P_{n-2}, n \geq 2;$$

and the Pell-Lucas sequence is defined by

$$Q_0 = 1, Q_1 = 1, Q_n = 2Q_{n-1} + Q_{n-2}, n \geq 2.$$

Notice that P_n and Q_n can be extended to integer indices by the recursion that gives

$$P_{-n} := (-1)^{n+1} P_n \text{ and } Q_{-n} := (-1)^{n+1} Q_n$$

for all $n \in \mathbb{N}_0$.

Lemma 2.1 ([8], pp. 148 – 149). *Let $k \in \mathbb{N}_0, n, m \in \mathbb{Z}$. Then we have:*

- a) $2P_{n+k}Q_k - (-1)^k P_n = P_{n+2k};$
- b) $P_{n+k}^2 - (-1)^n P_k^2 = P_{n+2k}P_n;$
- c) $P_{m+n} + (-1)^n P_{m-n} = 2P_mQ_n$ with $m \geq n$.

Now, we determine those shifts k, t and, s which generate an unbounded greatest common divisor function

$$G(n) := \gcd(P_n + s, P_{n+k} + t).$$

Then, we obtain the following result.

Theorem 2.1. *Let $s, t \in \mathbb{Z}$ and $k \in \mathbb{N}$. The following assertions on $G(n)$ are equivalent:*

- 1. $n \mapsto G(n)$ is an unbounded function on $\mathbb{N}_0$;
- 2. $(s, t) \in \{\pm(P_j, P_{j+k}) : j \in \mathbb{Z}\};$
- 3. $t^2 - 2Q_kst + (-1)^k s^2 = P_k^2$ or $-P_k^2$.

Proof. We give a proof of Theorem 2.1 by the implications $(1) \Rightarrow (3) \Rightarrow (2) \Rightarrow (1)$.

$(1) \Rightarrow (3)$: By Lemma 2.1. a) and b), we eliminate P_{n+2k} and get mod $G(n)$:

$$(-2Q_kt + (-1)^k s)(-s) \equiv t^2 - (-1)^n P_k^2 \pmod{G(n)}.$$

Set $e(n) = t^2 - 2Q_kst + (-1)^k s^2 - (-1)^n P_k^2$, then $e(n) \equiv 0 \pmod{G(n)}$. Since $e(n) = e(0)$ or $e(1)$ and has arbitrarily large divisors by (1), we get $e(0) = 0$ or $e(1) = 0$.

(3) $\Rightarrow$ (2): We have

$$\begin{aligned}\pm P_k^2 &= t^2 - 2Q_k st + (-1)^k s^2 \\ &= (t - Q_k s)^2 - (Q_k^2 - (-1)^k) s^2.\end{aligned}$$

From $Q_k^2 - (-1)^k = 2P_k^2$, we get $a := \frac{t - Q_k s}{P_k}$ is an integer and $a^2 - 2s^2 = \pm 1$. Hence, $a + s\sqrt{2}$ is a unit in the quadratic field $\mathbb{Q}(\sqrt{2})$. By [7, Theorem 244], there exists $j \in \mathbb{Z}$ such that

$$a + s\sqrt{2} = \pm \gamma^j, \text{ where } \gamma := 1 + \sqrt{2}.$$

From $Q_j + P_j\sqrt{2} = \gamma^j$, we can conclude that $s = \pm P_j$, $a = \pm Q_j$ and

$$t = aP_k + Q_k s = \pm(Q_j P_k + Q_k P_j) = \pm P_{j+k}.$$

Hence, $(s, t) = \pm(P_j, P_{j+k})$.

(2) $\Rightarrow$ (1): Let $n \equiv j \pmod{4}$ and set $u = \frac{n+j}{2}$, $v = \frac{n-j}{2}$ then v is even. By Lemma 2.1. c) we have

$$P_n + P_j = 2P_{\frac{n-j}{2}} Q_{\frac{n+j}{2}}$$

and by setting $u = \frac{n+j}{2} + k$, $v = \frac{n-j}{2}$, by Lemma 2.1. c) again, we also have

$$P_{n+k} + P_{j+k} = 2P_{\frac{n-j}{2}} Q_{\frac{n+j}{2}+k}.$$

This implies that $P_{\frac{n-j}{2}}$ is a common divisor of $P_n + P_j$ and $P_{n+k} + P_{j+k}$ if $n \equiv j \pmod{4}$. Hence, $\gcd(P_n + P_j, P_{n+k} + P_{j+k})$ takes arbitrarily large values.

Similarly, we can prove that the values $\gcd(P_n - P_j, P_{n+k} - P_{j+k})$ are unbounded. $\square$

It is well-known that the unboundedness of the function $G(n)$ implies the periodicity. Indeed, let $e(n) := t^2 - 2Q_k st + (-1)^k s^2 + (-1)^n P_k^2$.

Theorem 2.2. *If $s, t \in \mathbb{Z}$, $k \in \mathbb{N}$ and $c := |e(0)e(1)| \neq 0$, then $n \mapsto G(n)$ is simply periodic on $\mathbb{N}_0$, which means there exists a positive integer p such that $G(n+p) = G(n)$ for all $n \geq 0$. A period $p \leq c^2$ can be chosen such that*

$$P_p \equiv 0 \pmod{c} \text{ and } P_{p+1} \equiv 1 \pmod{c}.$$

Proof. The proof of this theorem is similar to that of [4]. From $c \neq 0$ and $G(n)$ divides c , there is a positive integer $p \leq c^2$ such that

$$P_n \equiv P_{n+p} \pmod{c} \text{ for all } n \in \mathbb{N}.$$

We have $G(n) = \gcd(P_n + s, P_{n+k} + t) \mid c$ and $P_{n+p} - P_n, P_{n+k+p} - P_{n+k}$ are divisible by c , then

$$G(n) \mid ((P_{n+p} - P_n) + (P_n + s)) \text{ and } G(n) \mid ((G_{n+k+p} - G_{n+k}) + (G_{n+k} + t)).$$

Thus,

$$G(n) \mid (P_{n+p} + s) \text{ and } G(n) \mid (P_{n+k+p} + t).$$

This implies $G(n) \mid \gcd(P_{n+k} + s, P_{n+p+k} + t)$. Hence, $G(n) \mid G(n + p)$. Similarly, we also have $G(n + p) \mid G(n)$. Therefore, $G(n + p) = G(n)$. $\square$

From

$$\begin{aligned} P_{j+1}^2 - 2P_j P_{j+1} - P_j^2 &= P_{j+1}(P_{j+1} - 2P_j) - P_j^2 \\ &= P_{j+1}P_{j-1} - P_j^2 \\ &= (-1)^j \end{aligned}$$

by Cassini - like formula, [8, p. 148], and for $k = 1$, we obtain a corollary of Theorem 2.1 as follows.

Theorem 2.3 ([9, Theorem 1]). *All solutions $(x, y) \in \mathbb{Z} \times \mathbb{Z}$ of*

1. $y^2 - 2xy - x^2 = 1$ *are of the form $(x, y) = \pm(P_j, P_{j+1}), j \in \mathbb{Z}$ and j is even;*
2. $y^2 - 2xy - x^2 = -1$ *are $(x, y) = \pm(P_j, P_{j+1}), j \in \mathbb{Z}$ and j is odd.*

In a similar way, one can prove the following Theorem 2.4 and Theorem 2.5 on unboundedness and periodicity of the Pell-Lucas sequence (Q_n) .

Theorem 2.4. *Let $s, t \in \mathbb{Z}, k \in \mathbb{N}$ and $G(n) := \gcd(Q_n + s, Q_{n+k} + t)$. The following assertions on $G(n)$ are equivalent:*

1. $n \mapsto G(n)$ *is an unbounded function on $\mathbb{N}_0$;*
2. $(s, t) \in \{\pm(Q_j, Q_{j+k}) : j \in \mathbb{Z}\}$;
3. $t^2 - 2Q_k st + (-1)^k s^2 = 2P_k^2$ *or* $-2P_k^2$.

$$\text{Let } e(n) := t^2 - 2Q_k st + (-1)^k s^2 + (-1)^n 2P_k^2.$$

Theorem 2.5. *If $s, t \in \mathbb{Z}; k \in \mathbb{N}$ and $c := |e(0)e(1)| \neq 0$, then $n \mapsto G(n)$ is simply periodic on $\mathbb{N}_0$, which means there exists a positive integer p such that $G(n + p) = G(n)$ for all $n \geq 0$. A period $p \leq c^2$ can be chosen such that*

$$Q_p \equiv 1 \pmod{c} \text{ and } Q_{p+1} \equiv 1 \pmod{c}.$$

As a consequence, we get the following theorem:

Theorem 2.6. *All solutions $(x, y) \in \mathbb{Z} \times \mathbb{Z}$ of*

1. $y^2 - 2xy - x^2 = 2$ *are of the form $(x, y) = \pm(Q_j, Q_{j+1}), j \in \mathbb{Z}$ and j is odd;*
2. $y^2 - 2xy - x^2 = -2$ *are of the form $(x, y) = \pm(Q_j, Q_{j+1}), j \in \mathbb{Z}$ and j is even.*

3. Fibonacci and Lucas sequences

The Fibonacci sequence $(F_n)_{n \geq 0}$ and Lucas sequence $(L_n)_{n \geq 0}$ are defined by the recursions

$$F_0 = 0, F_1 = 1; F_n = F_{n-1} + F_{n-2}, n \geq 2;$$

and

$$L_0 = 2, L_1 = 1; L_n = L_{n-1} + L_{n-2}, n \geq 2.$$

In this section, we recall the results presented in [5], [6] concerning the sequence on (F_n) and (L_n) . Those results are utilized to solve some interesting Diophantine equations in Theorem 3.3 and Theorem 3.5.

Theorem 3.1 ([6, Theorem 1]). *Let $s, t \in \mathbb{Z}, k \in \mathbb{N}$ and $G(n) := \gcd(F_n + s, F_{n+k} + t)$. The following assertions are equivalent:*

1. $n \mapsto G(n)$ is unbounded on $\mathbb{N}_0$;
2. $(s, t) \in \{\pm(F_j, F_{j+k}) : j \in \mathbb{Z}\}$;
3. $t^2 - L_k st + (-1)^k s^2 = F_k^2$ or $-F_k^2$.

$$\text{Let } e(n) := t^2 - L_k st + (-1)^k s^2 + (-1)^n F_k^2.$$

Theorem 3.2 ([6, Theorem 2]). *If $k \in \mathbb{N}, s, t \in \mathbb{Z}$ and $c := |e(0)e(1)| \neq 0$, then $n \mapsto G(n)$ is simply periodic on $\mathbb{N}_0$. A period $p \leq c^2$ can be chosen such that*

$$F_p \equiv 0 \pmod{c} \text{ and } F_{p+1} \equiv 1 \pmod{c}.$$

As a consequence, we have the following theorem:

Theorem 3.3. *All solutions $(x, y) \in \mathbb{Z} \times \mathbb{Z}$ of*

1. $y^2 - 2xy - x^2 = 1$ are of the form $(x, y) = \pm(F_j, F_{j+1}), j \in \mathbb{Z}$ and j is even;
2. $y^2 - 2xy - x^2 = -1$ are of the form $(x, y) = \pm(F_j, F_{j+1}), j \in \mathbb{Z}$ and j is odd.

The corresponding theorems for the Lucas sequence are as follows.

Theorem 3.4 ([6, Theorem 8]). *Let $k \in \mathbb{N}, s, t \in \mathbb{Z}$ and $G(n) := \gcd(L_n + s, L_{n+k} + t)$. The following assertions are equivalent:*

1. The function $n \mapsto G(n)$ is unbounded on $\mathbb{N}_0$;
2. $(s, t) \in \{\pm(L_j, L_{j+k}) : j \in \mathbb{Z}\}$;
3. $t^2 - L_k st + (-1)^k s^2 = 5F_k^2$ or $-5F_k^2$.

Theorem 3.5 ([9, Corollary 2]). *All solutions $(x, y) \in \mathbb{Z} \times \mathbb{Z}$ of*

1. $y^2 - 2xy - x^2 = 5$ are of the form $(x, y) = (L_j, L_{j+1}), j \in \mathbb{Z}$ and j is odd;
2. $y^2 - 2xy - x^2 = -5$ are of the form $(x, y) = (L_j, L_{j+1}), j \in \mathbb{Z}$ and j is even.

4. Balancing and Lucas – Balancing sequences

The balancing sequence $(B_n)_{n \geq 0}$ and the Lucas-balancing sequence $(C_n)_{n \geq 0}$ are defined by the recursions

$$B_0 = 0, B_1 = 1; B_n = 6B_{n-1} - B_{n-2}, n \geq 2;$$

and

$$C_0 = 1, C_1 = 3; C_n = 6C_{n-1} - C_{n-2}, n \geq 2.$$

By similar lines in the proofs as in Section 2., we also obtain the following results. The proofs of these results are omitted.

Theorem 4.1. *Let $s, t \in \mathbb{Z}, k \in \mathbb{N}$ and $G(n) := \gcd(B_n + s, B_{n+k} + t)$. The following assertions are equivalent:*

1. $n \mapsto G(n)$ is unbounded on $\mathbb{N}_0$;
2. $(s, t) \in \{\pm(B_j, B_{j+k}) : j \in \mathbb{Z}\}$;
3. $t^2 - 2C_k st + s^2 = B_k^2$.

Theorem 4.2 ([9, Theorem 2]). *All solutions $(x, y) \in \mathbb{Z} \times \mathbb{Z}$ of $y^2 - 6xy + x^2 = 1$ are of the form $(x, y) = \pm(B_j, B_{j+1}), j \in \mathbb{Z}$.*

Theorem 4.3. *Let $s, t \in \mathbb{Z}, k \in \mathbb{N}$ and $G(n) := \gcd(C_n + s, C_{n+k} + t)$. The following assertions are equivalent:*

1. $n \mapsto G(n)$ is unbounded on $\mathbb{N}_0$;
2. $(s, t) \in \{\pm(C_j, C_{j+k}) : j \in \mathbb{Z}\}$;
3. $t^2 - 2C_k st + s^2 = -8B_k^2$.

Theorem 4.4. *All solutions $(x, y) \in \mathbb{Z} \times \mathbb{Z}$ of $y^2 - 6xy + x^2 = -8$ are of the form $(x, y) = (C_j, C_{j+1}), j \in \mathbb{Z}$.*

Note for contributors:

- Short bio: Luu Ba Thang is a senior lecturer at the School of Mathematics and Computer Science, Hanoi National University of Education, Vietnam.

- Author's contributions: These results are entirely the author's own, from the initial concepts to the structuring, writing, proofs and editing.

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