

SEMI-IMPLICIT MILSTEIN SCHEME FOR STOCHASTIC GINZBURG-LANDAU EQUATIONS

Nguyen Viet Dung¹ and Nguyen Thi Kim Oanh²

¹*Dao Duy Tu High School, Thanh Hoa, Vietnam*

²*Xuan Phuong High School, Hanoi, Vietnam*

*Corresponding author: Nguyen Thi Kim Oanh, e-mail: oanhkim060601@gmail.com

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Abstract. We propose a semi-implicit Milstein scheme for a class of stochastic differential equations with super-linearly growing dissipative drift coefficients, in particular on the stochastic Ginzburg–Landau equation. The nonlinear cubic drift is treated implicitly, while the diffusion and Milstein correction terms are evaluated explicitly. This construction preserves the dissipative structure of the underlying equation and avoids the use of taming or truncation techniques. Under the assumption that the diffusion coefficient possesses bounded derivatives up to second order, we establish the well-posedness of the proposed numerical scheme and derive uniform moment bounds of arbitrary order. We further prove that the proposed approximation converges strongly to the exact solution with order one in L^2 . Numerical experiments confirm the theoretical convergence rate and demonstrate the stability of the method for large initial data.

Keywords: Ginzburg-Landau equations; Milstein scheme; Simulation; Stochastic differential equation.

1. Introduction

The numerical approximation of stochastic differential equations (SDEs) with non-globally Lipschitz coefficients has attracted considerable attention during the past two decades; see, for example, [1], [2]. Under global Lipschitz assumptions on both the drift and diffusion coefficients, classical numerical methods such as Euler–Maruyama (EM) scheme and the Milstein scheme are well understood. In particular, the EM method achieves a strong convergence order of $1/2$, while the Milstein method attains higher strong order 1 provided that sufficient smoothness assumptions on the diffusion coefficient are imposed [1], [3].

However, in many applications arising from physics, chemistry, biology, and finance, the drift coefficient exhibits super-linear growth and fails to satisfy the global Lipschitz condition. A typical example is the stochastic Ginzburg–Landau equation

$$dX_t = (aX_t - bX_t^3)dt + \sigma(X_t)dW_t, \quad 0 \leq t \leq T, \quad (1.1)$$

where $a \in \mathbb{R}$, $b > 0$, $\sigma(\cdot)$ is a sufficiently smooth diffusion coefficient, W_t is a standard one-dimensional Brownian motion defined on a complete filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})$. Owing to its nonlinear dissipative structure and rich dynamical behavior, equation (1.1) has become a standard benchmark problem in the theory of SDEs and numerical stochastic analysis. The cubic drift term provides strong stabilization for large values of the solution, but simultaneously creates significant difficulties for explicit numerical schemes.

It is now well known that the classical EM method may diverge strongly and weakly for SDEs with super-linearly growing drift coefficients, even when the exact solution possesses finite moments of all orders [4]. To overcome the divergence of the classical Euler–Maruyama method in the non-globally Lipschitz setting, a variety of modified explicit schemes have been developed. Among the most influential approaches are the tamed Euler methods introduced by Hutzenthaler, Jentzen and Kloeden [5] and further developed by Sabanis [6], as well as truncated Euler methods proposed by Mao [7] and later generalized in [8]. Adaptive step-size techniques and stopping-time arguments have also been investigated; see, for example, [9], [10].

Beyond Euler-type methods, considerable effort has been devoted to the construction of higher-order approximations. Under global Lipschitz assumptions, the Milstein scheme is one of the most important strong approximation methods because it achieves convergence order one in L^2 while retaining a relatively simple structure [1], [3]. Extending Milstein-type methods to non-globally Lipschitz SDEs is considerably more challenging. Several modified Milstein schemes have been proposed in recent years, including tamed Milstein methods [11]–[13] and related variants for super-linear stochastic systems. These methods preserve the higher-order structure of the Milstein correction while introducing suitable modifications of the drift and diffusion coefficients to guarantee stability and convergence. An alternative strategy is provided by implicit discretization. Implicit methods are well known for their superior stability properties and their ability to preserve dissipative structures of stochastic dynamical systems; see [14]–[16]. In particular, they are especially effective for stiff equations and SDEs with strongly nonlinear drift coefficients. The drawback is that fully implicit schemes typically require the solution of nonlinear algebraic equations involving the entire drift coefficient at every time step, which may become computationally demanding.

Recently, in [17], a semi-implicit Euler–Maruyama scheme was proposed for stochastic differential equations whose drift can be decomposed into a globally Lipschitz component and a super-linear dissipative component. The key idea is to treat only the

nonlinear dissipative part implicitly while keeping the remaining terms explicit. This strategy preserves much of the stability of fully implicit methods while significantly reducing the computational complexity. Moreover, the resulting numerical scheme retains the original drift structure and avoids taming or truncation. Under suitable monotonicity and Khasminskii-type assumptions, the scheme was shown to possess bounded moments and to converge strongly with order $1/2$.

The purpose of the present paper is therefore to develop and analyze a semi-implicit Milstein scheme for the stochastic Ginzburg–Landau equation (1.1). The proposed method combines the stabilizing effect of an implicit treatment of the cubic drift with the higher-order accuracy of the Milstein correction. More precisely, the nonlinear term $-bX_t^3$ is discretized implicitly, whereas the diffusion coefficient and Milstein correction are evaluated explicitly. As a consequence, the numerical method leads to a simple scalar nonlinear equation at each time step while preserving the dissipative structure of the original system. More precisely, we introduce a semi-implicit Milstein scheme of the form

$$Y_{k+1} = Y_k + (aY_k - bY_{k+1}^3)\Delta + \sigma(Y_k)\Delta W_k + \frac{1}{2}\sigma(Y_k)\sigma'(Y_k)((\Delta W_k)^2 - \Delta), \quad (1.2)$$

where $\Delta = T/n$, $\Delta W_k = W_{(k+1)\Delta} - W_{k\Delta}$. Compared with the fully implicit Milstein method, the proposed scheme leads to a significantly simpler nonlinear equation at each time step. At the same time, it preserves the dissipative structure generated by the cubic term $-bX^3$, which plays a crucial role for obtaining uniform moment estimates.

The present work can be regarded as a higher-order extension of the semi-implicit Euler–Maruyama approach recently developed in [17] for super-linear stochastic differential equations. Compared with the semi-implicit Euler method, the analysis here is substantially more delicate because of the presence of the Milstein correction term $\frac{1}{2}\sigma(X)\sigma'(X)((\Delta W)^2 - \Delta)$, whose contribution must be controlled simultaneously with the implicit cubic drift. The combination of these two features constitutes the main technical difficulty of the paper.

2. Uniform moment estimates

Throughout this paper, we suppose that the following assumption holds.

Assumption 2.1. *The initial value X_0 is bounded, $b > 0$, the function $\sigma \in C^2(\mathbb{R})$, and there exists a constant $L > 0$ such that*

$$|\sigma'(x)| + |\sigma''(x)| \leq L, \quad x \in \mathbb{R}.$$

Thanks to Assumption 2.1, there exists a constant $C > 0$ such that

$$|\sigma(x)| + |h(x)| \leq C(1 + |x|), \quad x \in \mathbb{R}. \quad (2.1)$$

It follows from Lemma 3.1 in [17] that for any $p > 0$, there exists a constant C_p such that

$$\mathbb{E}\left[\sup_{0 \leq t \leq T} |X_t|^p\right] < C_p.$$

We rewrite the semi-implicit Milstein scheme as follows,

$$Y_{k+1} = Y_k + (aY_k - bY_{k+1}^3)\Delta + \sigma(Y_k)\Delta W_k + h(Y_k)\eta_k, \quad (2.2)$$

where

$$Y_0 = X_0, \quad h(x) = \frac{1}{2}\sigma(x)\sigma'(x), \quad \eta_k = (\Delta W_k)^2 - \Delta.$$

It is clear that there exists a unique Y_{k+1} satisfying equation (2.2).

Lemma 2.1. *For every integer $m \geq 1$, there exists a constant $C_m > 0$, independent of Δ and n , such that*

$$\sup_{0 \leq k \leq n} \mathbb{E}|Y_k|^{2m} \leq C_m.$$

Proof. Rewrite (2.2) as

$$Y_{k+1} + b\Delta Y_{k+1}^3 = Y_k + Z_k, \quad (2.3)$$

where $Z_k = a\Delta Y_k + \sigma(Y_k)\Delta W_k + h(Y_k)\eta_k$. Multiplying both sides of (2.3) by Y_{k+1}^{2m-1} gives $Y_{k+1}^{2m} + b\Delta Y_{k+1}^{2m+2} = Y_{k+1}^{2m-1}(Y_k + Z_k)$. Since $b > 0$,

$$|Y_{k+1}|^{2m} \leq |Y_k + Z_k|^{2m}. \quad (2.4)$$

We now estimate the right-hand side conditionally on $\mathcal{F}_{t_k}$. By the binomial formula,

$$(Y_k + Z_k)^{2m} = Y_k^{2m} + 2mY_k^{2m-1}Z_k + \sum_{r=2}^{2m} \binom{2m}{r} Y_k^{2m-r} Z_k^r.$$

Taking conditional expectation and using $\mathbb{E}[\Delta W_k | \mathcal{F}_{t_k}] = 0$, $\mathbb{E}[\eta_k | \mathcal{F}_{t_k}] = 0$, we obtain $\mathbb{E}[Z_k | \mathcal{F}_{t_k}] = a\Delta Y_k$. Consequently,

$$2mY_k^{2m-1}\mathbb{E}[Z_k | \mathcal{F}_{t_k}] = 2ma\Delta Y_k^{2m} \leq C\Delta|Y_k|^{2m}.$$

Next, for every integer $r \geq 2$, using (2.1) and the moment estimates of Gaussian increments, we have

$$\mathbb{E}[|Z_k|^r | \mathcal{F}_{t_k}] \leq C_r \Delta^{r/2} (1 + |Y_k|^r) \leq C_r \Delta (1 + |Y_k|^r), \quad (2.5)$$

where we used $\Delta \leq T$ and $r/2 \geq 1$. Indeed, $\mathbb{E}|\Delta W_k|^r = C_r \Delta^{r/2}$, and $\mathbb{E}|\eta_k|^r = \mathbb{E}|(\Delta W_k)^2 - \Delta|^r \leq C_r \Delta^r \leq C_r \Delta^{r/2}$. Therefore (2.5) follows from

$$|Z_k|^r \leq C_r \left(\Delta^r |Y_k|^r + |\sigma(Y_k)|^r |\Delta W_k|^r + |h(Y_k)|^r |\eta_k|^r \right).$$

For $2 \leq r \leq 2m$, Young's inequality gives $|Y_k|^{2m-r}(1 + |Y_k|^r) \leq C_m(1 + |Y_k|^{2m})$. Thus $\mathbb{E}[|Y_k|^{2m-r}|Z_k|^r \mid \mathcal{F}_{t_k}] \leq C_m\Delta(1 + |Y_k|^{2m})$. Combining the above estimates with (2.4), we obtain

$$\mathbb{E}[|Y_{k+1}|^{2m} \mid \mathcal{F}_{t_k}] \leq (1 + C_m\Delta)|Y_k|^{2m} + C_m\Delta.$$

Taking expectation yields

$$\mathbb{E}|Y_{k+1}|^{2m} \leq (1 + C_m\Delta)\mathbb{E}|Y_k|^{2m} + C_m\Delta.$$

By the discrete Gronwall inequality, $\sup_{0 \leq k \leq n} \mathbb{E}|Y_k|^{2m} \leq C_m$. The proof is complete. $\square$

3. Strong convergence of order one

Let $f(x) = ax - bx^3$. For $0 \leq k \leq n - 1$, define

$$\begin{aligned} \rho_{k+1} := & \int_{t_k}^{t_{k+1}} [f(X(s)) - aX(t_k) + bX(t_{k+1})^3] ds \\ & + \int_{t_k}^{t_{k+1}} \left[\sigma(X(s)) - \sigma(X(t_k)) - \sigma(X(t_k))\sigma'(X(t_k))(W(s) - W(t_k)) \right] dW(s). \end{aligned} \quad (3.1)$$

Lemma 3.1. *There exists a constant $C > 0$, independent of k and Δ , such that*

$$\mathbb{E}|\rho_{k+1}|^2 \leq C\Delta^3. \quad (3.2)$$

Moreover,

$$\mathbb{E}|\mathbb{E}[\rho_{k+1} \mid \mathcal{F}_{t_k}]|^2 \leq C\Delta^4, \quad (3.3)$$

$$\mathbb{E}|\mathbb{E}[\rho_{k+1}\Delta W_k \mid \mathcal{F}_{t_k}]|^2 \leq C\Delta^4, \quad (3.4)$$

$$\mathbb{E}|\mathbb{E}[\rho_{k+1}\eta_k \mid \mathcal{F}_{t_k}]|^2 \leq C\Delta^4. \quad (3.5)$$

Proof. For simplicity, write $X_k = X(t_k)$, $X_{k+1} = X(t_{k+1})$, $\mathbb{E}_k[\cdot] = \mathbb{E}[\cdot \mid \mathcal{F}_{t_k}]$. We decompose $\rho_{k+1} = A_{k+1} + B_{k+1}$, where

$$\begin{aligned} A_{k+1} &= \int_{t_k}^{t_{k+1}} [f(X(s)) - aX_k + bX_{k+1}^3] ds \\ &= a \int_{t_k}^{t_{k+1}} (X(s) - X_k) ds - b \int_{t_k}^{t_{k+1}} (X(s)^3 - X_{k+1}^3) ds, \end{aligned}$$

and

$$B_{k+1} = \int_{t_k}^{t_{k+1}} \left[\sigma(X(s)) - \sigma(X_k) - \sigma(X_k)\sigma'(X_k)(W(s) - W(t_k)) \right] dW(s).$$

We shall use the standard conditional increment estimates: for every $p \geq 2$, there exist constants $C_p > 0$ such that, for $t_k \leq s \leq t_{k+1}$,

$$\mathbb{E}_k |X(s) - X_k|^p \leq C_p(1 + |X_k|^{3p})(s - t_k)^{p/2}, \quad (3.6)$$

and

$$\mathbb{E}_k |X(s) - X_k - \sigma(X_k)(W(s) - W(t_k))|^2 \leq C(1 + |X_k|^6)(s - t_k)^2. \quad (3.7)$$

These estimates follow from the integral form of the SDE, the Burkholder–Davis–Gundy inequality and the polynomial moment bounds for the exact solution.

Hölder’s inequality, (3.6), and the moment bounds for X yield

$$\begin{aligned} \mathbb{E}|A_{k+1}|^2 &\leq C\Delta \int_{t_k}^{t_{k+1}} \mathbb{E}|X(s) - X_k|^2 ds \\ &\quad + C\Delta \int_{t_k}^{t_{k+1}} \mathbb{E}[(1 + |X(s)|^4 + |X_{k+1}|^4)|X(s) - X_{k+1}|^2] ds \\ &\leq C\Delta \int_{t_k}^{t_{k+1}} (s - t_k) ds + C\Delta \int_{t_k}^{t_{k+1}} (t_{k+1} - s) ds \leq C\Delta^3. \end{aligned}$$

We now estimate the diffusion defect. Define

$$R(s) = \sigma(X(s)) - \sigma(X_k) - \sigma(X_k)\sigma'(X_k)(W(s) - W(t_k)).$$

By Itô’s formula applied to $\sigma(X(s))$,

$$\begin{aligned} R(s) &= \int_{t_k}^s \left[\sigma'(X(u))f(X(u)) + \frac{1}{2}\sigma''(X(u))\sigma(X(u))^2 \right] du \\ &\quad + \int_{t_k}^s \left[\sigma'(X(u))\sigma(X(u)) - \sigma'(X_k)\sigma(X_k) \right] dW(u). \end{aligned}$$

Since σ' and σ'' are bounded, using (3.6),

$$\mathbb{E}_k |R(s)|^2 \leq C(1 + |X_k|^6)(s - t_k)^2.$$

Therefore, by Itô’s isometry,

$$\mathbb{E}|B_{k+1}|^2 = \int_{t_k}^{t_{k+1}} \mathbb{E}|R(s)|^2 ds \leq C \int_{t_k}^{t_{k+1}} (s - t_k)^2 ds \leq C\Delta^3.$$

Combining the estimates for A_{k+1} and B_{k+1} proves (3.2).

We now prove the conditional estimates. Since B_{k+1} is a stochastic integral over $[t_k, t_{k+1}]$ with an adapted integrand, $\mathbb{E}_k[B_{k+1}] = 0$. Thus $\mathbb{E}_k[\rho_{k+1}] = \mathbb{E}_k[A_{k+1}]$. Using Itô’s formula for $X(s)$ and $X(s)^3$, one obtains

$$|\mathbb{E}_k[X(s) - X_k]| \leq C \int_{t_k}^{t_{k+1}} (1 + \mathbb{E}_k[|X_r|^3]) dr.$$

and

$$|\mathbb{E}_k[X(s)^3 - X_{k+1}^3]| \leq C \int_{t_k}^{t_{k+1}} (1 + \mathbb{E}_k[|X_r|^5]) dr.$$

Consequently,

$$|\mathbb{E}_k[A_{k+1}]| \leq C\Delta \int_{t_k}^{t_{k+1}} (1 + \mathbb{E}_k[|X_s|^5]) ds.$$

Taking expectation and using the moment bounds of X_k gives $\mathbb{E} |\mathbb{E}_k[\rho_{k+1}]|^2 \leq C\Delta^4$. This proves (3.3).

The estimates (3.4) and (3.5) follow directly from (3.2) and the Cauchy-Schwarz inequality. $\square$

Theorem 3.1. *Let assumption 2.1 hold. Then, there exists a constant $C > 0$, independent of Δ , such that*

$$\max_{0 \leq k \leq n} \mathbb{E}|X(t_k) - Y_k|^2 \leq C\Delta^2.$$

Thus, the semi-implicit Milstein scheme converges strongly with order one in L^2 .

Proof. Let $e_k = X(t_k) - Y_k$. It follows from (2.2) that

$$e_{k+1} + b\Delta(X(t_{k+1})^3 - Y_{k+1}^3) = (1 + a\Delta)e_k + \delta\sigma_k\Delta W_k + \delta h_k\eta_k + \rho_{k+1}, \quad (3.8)$$

where ρ_{k+1} is defined in (3.1) and

$$\delta\sigma_k = \sigma(X(t_k)) - \sigma(Y_k), \quad \delta h_k = h(X(t_k)) - h(Y_k).$$

Define $U_{k+1} = (1 + a\Delta)e_k + \delta\sigma_k\Delta W_k + \delta h_k\eta_k$. Then

$$e_{k+1} + b\Delta(X(t_{k+1})^3 - Y_{k+1}^3) = U_{k+1} + \rho_{k+1}.$$

Multiplying both sides by e_{k+1} gives

$$|e_{k+1}|^2 + b\Delta e_{k+1}(X(t_{k+1})^3 - Y_{k+1}^3) = e_{k+1}(U_{k+1} + \rho_{k+1}).$$

Since $(x - y)(x^3 - y^3) = (x - y)^2(x^2 + xy + y^2) \geq 0$, the second term on the left-hand side is nonnegative. Hence

$$|e_{k+1}|^2 \leq e_{k+1}(U_{k+1} + \rho_{k+1}).$$

By Young's inequality, $|e_{k+1}|^2 \leq |U_{k+1} + \rho_{k+1}|^2$. Therefore,

$$\mathbb{E}|e_{k+1}|^2 \leq \mathbb{E}|U_{k+1}|^2 + 2\mathbb{E}[U_{k+1}\rho_{k+1}] + \mathbb{E}|\rho_{k+1}|^2. \quad (3.9)$$

We have

$$\mathbb{E}|U_{k+1}|^2 = (1 + a\Delta)^2\mathbb{E}|e_k|^2 + \Delta\mathbb{E}|\delta\sigma_k|^2 + 2\Delta^2\mathbb{E}|\delta h_k|^2 \leq (1 + C\Delta)\mathbb{E}|e_k|^2, \quad (3.10)$$

where we use the global Lipschitz continuity of σ and h . We write $\mathbb{E}[U_{k+1}\rho_{k+1}] = I_1 + I_2 + I_3$, with

$$I_1 = \mathbb{E}\left[(1 + a\Delta)e_k \rho_{k+1}\right], \quad I_2 = \mathbb{E}\left[\delta\sigma_k \Delta W_k \rho_{k+1}\right], \quad I_3 = \mathbb{E}\left[\delta h_k \eta_k \rho_{k+1}\right].$$

For the first term, using conditional expectation and the Cauchy–Schwarz inequality,

$$|I_1| = \left| \mathbb{E}\left[(1 + a\Delta)e_k \mathbb{E}_k[\rho_{k+1}]\right] \right| \leq C(\mathbb{E}|e_k|^2)^{1/2} \left(\mathbb{E}|\mathbb{E}_k[\rho_{k+1}]|^2 \right)^{1/2}.$$

By (3.3), $|I_1| \leq C\Delta^2(\mathbb{E}|e_k|^2)^{1/2} \leq C\Delta \mathbb{E}|e_k|^2 + C\Delta^3$. For the second term, since $\delta\sigma_k$ is $\mathcal{F}_{t_k}$ -measurable, $I_2 = \mathbb{E}\left[\delta\sigma_k \mathbb{E}_k[\rho_{k+1}\Delta W_k]\right]$. Since σ is globally Lipschitz, $|\delta\sigma_k| \leq L|e_k|$. Therefore,

$$|I_2| \leq L\mathbb{E}\left[|e_k| |\mathbb{E}_k[\rho_{k+1}\Delta W_k]|\right] \leq L(\mathbb{E}|e_k|^2)^{1/2} \left(\mathbb{E}|\mathbb{E}_k[\rho_{k+1}\Delta W_k]|^2 \right)^{1/2}.$$

Using (3.4), $|I_2| \leq C\Delta^2(\mathbb{E}|e_k|^2)^{1/2} \leq C\Delta \mathbb{E}|e_k|^2 + C\Delta^3$. Since h is globally Lipschitz, $|\delta h_k| \leq L|e_k|$, and $I_3 = \mathbb{E}\left[\delta h_k \mathbb{E}_k[\rho_{k+1}\eta_k]\right]$. Hence

$$|I_3| \leq L(\mathbb{E}|e_k|^2)^{1/2} \left(\mathbb{E}|\mathbb{E}_k[\rho_{k+1}\eta_k]|^2 \right)^{1/2}.$$

By (3.5), $|I_3| \leq C\Delta^2(\mathbb{E}|e_k|^2)^{1/2} \leq C\Delta \mathbb{E}|e_k|^2 + C\Delta^3$. Combining the above estimates, we arrive at

$$\left| \mathbb{E}[U_{k+1}\rho_{k+1}] \right| \leq C\Delta \mathbb{E}|e_k|^2 + C\Delta^3. \quad (3.11)$$

Substituting (3.11), (3.10), and (3.2) into (3.9), we obtain

$$\mathbb{E}|e_{k+1}|^2 \leq (1 + C\Delta)\mathbb{E}|e_k|^2 + C\Delta^3.$$

Since $e_0 = 0$, an application of discrete Gronwall inequality yields

$$\max_{0 \leq k \leq n} \mathbb{E}|e_k|^2 \leq C\Delta^2.$$

This completes the proof. □

4. Numerical study

In this section, we present some numerical experiments to illustrate the convergence behavior of the proposed semi-implicit Milstein scheme. The experiments are conducted to verify the theoretical strong convergence order established in Theorem 3.1.

We consider the stochastic Ginzburg–Landau equation

$$dX_t = (aX_t - bX_t^3)dt + \sigma(X_t)dW_t, \quad X_0 = x_0, \quad (4.1)$$

with $a = -1$, $b = 1$, $T = 1$. Two diffusion coefficients are considered: $\sigma(x) = x + 1$, and $\sigma(x) = \cos(x)$. For both two cases, the assumptions of Theorem 3.1 are satisfied since σ' , σ'' are globally bounded or have bounded derivatives.

Since the exact solution is unavailable in closed form, we employ the standard two-scale strategy used in strong convergence studies. For each step size $\Delta_k = 2^{-k}$, $k = 4, \dots, 10$, we compute a coarse approximation $Y^{\Delta_k}(T)$ and a fine approximation $Y^{\Delta_k/2}(T)$, driven by the same Brownian path. The root mean square difference (RMSD) is defined by

$$\text{RMSD}(\Delta_k) = \left(\frac{1}{M} \sum_{j=1}^M \left| Y_j^{\Delta_k}(T) - Y_j^{\Delta_k/2}(T) \right|^2 \right)^{1/2},$$

where M denotes the number of Monte Carlo samples.

We estimate the convergence rate α from the least-squares fit

$$\log_2(\text{RMSD}(\Delta_k)) = -\alpha k + \beta.$$

If the numerical method converges strongly with order one, then we expect $\alpha \approx 1$. Throughout the experiments, we use $M = 10^5$ Monte Carlo trajectories.

Table 1: Estimated strong convergence rates of the semi-implicit Milstein scheme for different diffusion coefficients.

X_0	0.1	0.5	1	2	3	5	10	20	50
$\sigma(x) = x + 1$	0.99	0.99	0.98	0.98	0.98	0.96	0.93	0.90	0.88
$\sigma(x) = \cos(x)$	1.00	1.00	1.00	0.99	0.98	0.96	0.93	0.91	0.90

Table 1 reports the estimated convergence rates for different initial values. We observe that the estimated rates remain close to one for all tested initial values. Even for relatively large initial conditions such as $X_0 = 50$, the estimated rate remains above 0.87. Moreover, all regression coefficients satisfy $R^2 > 0.999$, indicating an excellent linear relation between $\log_2(\text{RMSD})$ and the refinement level k .

These results strongly support the theoretical convergence order predicted by Theorem 3.1.

Note for contributor.

- Short bio: Nguyen Viet Dung is a mathematics teacher at Dao Duy Tu High School in Thanh Hoa. Nguyen Thi Kim Oanh is a mathematics teacher at Xuan Phuong High School, Hanoi.

- Author's contributions: Nguyen Thi Kim Oanh carried out the theoretical analysis, including the derivation of moment estimates and the proof of the strong convergence theorem; Nguyen Viet Dung developed the numerical implementation, designed and performed the computational experiments, and analyzed the numerical results. Nguyen Thi Kim Oanh and Nguyen Viet Dung jointly interpreted the theoretical and numerical findings, contributed to the writing and revision of the manuscript, and approved the final version for publication.

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