

STUDY OF THE EXISTENCE AND STABILITY OF SPATIAL SOLITON BEAMS IN HIGHER-ORDER NONLINEAR OPTICAL FIBERS WITH GAIN AND LOSS

Nguyen Viet Hung

Hanoi University of Science and Technology

Corresponding author: Nguyen Viet Hung, e-mail: hung.nguyenviet1@hust.edu.vn

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Abstract. In this paper, we employ an integrated numerical procedure for computing spatial soliton beams, or stable nonlinear guided modes, in non-Hermitian photonic waveguides. The physical systems considered here are higher-order nonlinear optical fibers with step-index (SI) and graded-index (GRIN) profiles. The fiber material exhibits appreciable two-photon absorption, which acts as a nonlinear loss mechanism. To obtain stable guided modes in these fibers, external pumping is introduced to provide linear gain. Starting from prescribed input optical beams, we obtain stable fundamental and vortex beams after sufficiently long propagation in the nonlinear fibers. These nonlinear dissipative modes are formed through a balance among diffraction, nonlinear effects, gain, and loss. The numerical results show that fundamental beams can be stabilized in both SI and GRIN fibers, whereas stable vortex beams are supported only in the GRIN fiber. The numerical contribution is the coordinated three-stage continuation procedure rather than any of the individual algorithms: conservative stationary seeds are first generated, gain and nonlinear loss are then introduced adiabatically, and the resulting dissipative states are finally validated by long-distance propagation with fixed parameters, including perturbation tests. The different vortex behavior in the two profiles indicates that the smooth radial confinement of the GRIN potential is important for preserving the annular vortex structure, whereas the flat SI core does not provide the same distributed restoring confinement.

Keywords: Nonlinear optical fiber, higher-order nonlinear medium, dissipative solitons, nonlinear Schrödinger equation, numerical methods.

1. Introduction

Nonlinear optical fibers play a crucial role in modern photonics by enabling phenomena such as soliton formation, supercontinuum generation, and all-optical signal processing [1], [2]. They are widely used in scientific research to investigate complex light-matter interactions and to develop advanced models of nonlinear wave

propagation. In practical applications, nonlinear fibers are essential for high-speed optical communication systems, optical switching, and emerging photonic technologies [1]-[4].

Third-order nonlinear materials are the most widely used media in photonics because they provide a strong and versatile nonlinear response in many optical systems. In recent years, higher-order nonlinear materials, including fifth- and seventh-order media, have also attracted increasing attention. These materials are being explored extensively because of their considerable potential for advanced applications in ultrafast optics and nonlinear wave control [5]-[10].

Nonlinear optical materials exhibiting multiphoton absorption, such as two-photon and three-photon absorption, enable processes in which multiple photons are absorbed simultaneously, typically at high optical intensities [11], [12]. Their key characteristics include strong intensity dependence, large nonlinear coefficients, and operation in spectral regions where linear absorption is weak. However, such materials also suffer from increased optical losses, heat generation, and reduced device efficiency, which may limit their practical performance.

To overcome these limitations, additional energy is often supplied to absorptive systems to compensate for losses and sustain optical processes. This requirement has motivated the development of photonic platforms based on non-Hermitian physics, where gain and loss are deliberately engineered [13]-[18]. Non-Hermitian photonic devices exhibit unique features such as exceptional points, unidirectional propagation, and enhanced sensitivity [13]-[16]. Consequently, they open new possibilities for controlling light in advanced optical communications, sensing, and signal-processing applications.

In such systems, finding stable propagating nonlinear modes is highly challenging because of the complex interplay among gain, loss, diffraction, and nonlinearities. The stable waveforms that can exist under these conditions are known as dissipative solitons [11]-[12], [19]-[22]. Their formation requires a precise balance not only between dispersion, or diffraction, and nonlinearity, but also between energy gain and dissipation. Therefore, the conditions for their existence are much more restrictive than in conventional Hermitian systems.

In this work, we present a numerical simulation study of the formation of stable propagating nonlinear modes in higher-order nonlinear optical fibers, including nonlinear contributions up to the seventh order. We consider both step-index and graded-index fiber profiles and include two-photon absorption as a source of nonlinear dissipation. To compensate for these losses and enable the existence of dissipative solitons, external energy is continuously supplied along the propagation direction, thereby introducing linear gain.

SI and GRIN profiles represent two standard and physically distinct guiding architectures. An SI fiber has an approximately uniform core index and an abrupt core-cladding interface, whereas a GRIN fiber has a continuously varying radial index that provides distributed transverse confinement and, in multimode propagation, reduces intermodal delay. Comparing the two profiles therefore allows us to isolate how the shape of the linear guiding potential affects the formation and stability of nonlinear localized modes [1], [23]-[26].

Common numerical methods developed for nonlinear Hermitian physical systems are not directly applicable to this problem. The search for stable nonlinear guided modes in these fibers is therefore challenging. By using an integrated numerical procedure that combines several computational techniques, we systematically analyze the existence and stability of nonlinear modes in the fibers. In particular, we focus on both fundamental and topological soliton beams, namely optical vortices [27], that can propagate in the two types of fibers. By exploring a broad range of input parameters, we determine the conditions under which these localized stable states can emerge and persist.

Related dissipative-soliton problems in higher-order nonlinear fibers have previously been studied, for example, by variational approximations and direct simulations in cubic-quintic fiber-laser models with multiphoton absorption/emission [11], and by linear-stability analysis together with numerical simulations for cubic-quintic-septic complex Ginzburg-Landau models [12]. Standard stationary solvers such as AITM, SOM, and NCG are also well established for conservative nonlinear-wave equations [28]-[31]. The present contribution is therefore not a new form of any of these algorithms; rather, it is their coordinated use in a continuation strategy tailored to the two-dimensional non-Hermitian SI/GRIN problem, including the search for both fundamental and vortex states.

The paper is organized as follows. Section 2 introduces physical models of non-Hermitian nonlinear fibers with step-index and graded-index profiles and then presents the integrated numerical procedure for obtaining stable nonlinear guided modes propagating in such fibers. Section 3 presents numerical simulation results for the generation of stable fundamental and topological guided modes in the fibers. The Conclusions summarize the main findings of this work.

2. The model and numerical method

We consider higher-order nonlinear optical fibers with nonlinear absorption and linear-gain pumping provided by an external laser source incident perpendicular to the fiber axis. The wave equation describing nonlinear modes in the fiber takes the form of a nonlinear Schrödinger equation [2], [12]

$$2ik \frac{\partial A}{\partial \tilde{z}} + \tilde{\nabla}^2 A + 2k^2 \left[\delta n(\tilde{\mathbf{r}}) + n_{NL} |A|^2 \right] A + ig A = 0 \quad (1)$$

Here, i is the imaginary unit, $A(\tilde{\mathbf{r}}, \tilde{z})$ denotes the slowly varying complex envelope of the laser-beam electric field, k is the wavenumber, δn is the refractive-index contrast between the fiber core and cladding, n_{NL} is the nonlinear refractive-index coefficient of the fiber material, g is the parameter characterizing the rate of energy pumping into the fiber along the axial direction, and the variables $\{\tilde{\mathbf{r}}, \tilde{z}\} = \{\tilde{x}, \tilde{y}, \tilde{z}\}$ are spatial coordinates in physical units.

We investigate two types of optical fibers, treated as cylindrically symmetric two-dimensional waveguides: an SI fiber and a GRIN fiber. Here, $\delta n(\tilde{\mathbf{r}}) = n(\tilde{\mathbf{r}}) - n_{clad}$, and the spatial distribution of the refractive-index function $n(\tilde{\mathbf{r}})$ is expressed for the SI fiber as

$$n(\tilde{\mathbf{r}}) = n(\tilde{r}) = \begin{cases} n_{core}, & 0 \leq \tilde{r} < \tilde{r}_c \\ n_{clad}, & \tilde{r} \geq \tilde{r}_c \end{cases} \quad (2)$$

where $\tilde{r} = |\tilde{\mathbf{r}}|$ and $\tilde{r}_c$ is the fiber-core radius and n_{core} , n_{clad} are the linear refractive indices of the core and cladding, respectively. For the GRIN fiber, the refractive index is approximately described by the following continuous function [1], [23]-[26]

$$n(\tilde{r}) = \begin{cases} n_{core} \left(1 - \frac{\tilde{r}^2}{\tilde{r}_c^2} \right) + n_{clad}, & 0 \leq \tilde{r} < \tilde{r}_c \\ n_{clad}, & \tilde{r} \geq \tilde{r}_c \end{cases} \quad (3)$$

The nonlinear refractive index of the fiber material depends on the intensity of the propagating beam according to

$$n_{NL}(|A|^2) = (n_{2R} + in_{2I})|A|^2 + n_4|A|^4 + n_6|A|^6 \quad (4)$$

Here, the quantity n_{2R} is the real cubic (Kerr) coefficient, n_{2I} is its imaginary part and represents nonlinear absorption, while n_4 and n_6 are the real fifth- and seventh-order refractive nonlinearities, respectively. Thus, the $n_4|A|^4$ and $n_6|A|^6$ contributions describe conservative higher-order refraction in the present model. It should be noted that nonlinear absorption in the material is included here, with the two-photon process, corresponding to the term n_{2I} , being regarded as the dominant contribution [11], [12]. Higher-order multiphoton absorption would require corresponding imaginary higher-order coefficients and is not considered here.

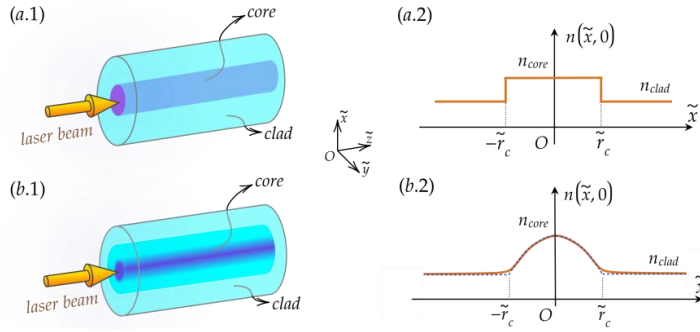


Figure 1. Models of SI (a1) and GRIN (b1) nonlinear optical fibers

The right panels (a2) and (b2) show the corresponding refractive-index distributions.

The schematics are not drawn to scale; the apparent core-size difference between panels (a1) and (b1) is illustrative only.

To normalize the propagation equation (1), we introduce the new variables and field function $r = \tilde{r} / \tilde{r}_c$; $z = \tilde{z} / (k \tilde{r}_c^2)$; $\Psi = A k \tilde{r}_c \sqrt{n_{2R}}$. Substituting these variables and the field transformation into the propagation equation yields

$$i \frac{\partial \Psi}{\partial z} = -\frac{1}{2} \nabla^2 \Psi - k^2 \tilde{r}_c^2 \delta n(r) \Psi - \frac{n_{2R} + in_{2I}}{n_{2R}} |\Psi|^2 \Psi - \frac{n_4}{k^2 \tilde{r}_c^2 n_{2R}} |\Psi|^4 \Psi - \frac{n_6}{k^4 \tilde{r}_c^4 n_{2R}^3} |\Psi|^6 \Psi - i \frac{g \tilde{r}_c^2}{2} \Psi \quad (5)$$

The dimensionless parameters are defined as follows:

$$V_{core} = -k^2 \tilde{r}_c^2 (n_{core} - n_{clad}), \quad (6)$$

$$V_{clad} = 0, \quad (7)$$

$$\gamma = -g \tilde{r}_c^2 / 2, \quad (8)$$

$$\Gamma = n_{2I} / n_{2R}, \quad (9)$$

$$\sigma = -n_4 / (k^2 \tilde{r}_c^2 n_{2R}^2), \quad (10)$$

$$\lambda = -n_6 / (k^4 \tilde{r}_c^4 n_{2R}^3). \quad (11)$$

It should be noted that the parameter γ can be readily tuned by changing the pump power of the external excitation source incident on the optical fiber. The parameters Γ , σ and λ denote the nonlinear absorption, quintic and septic coefficients, respectively. The last three normalized nonlinear parameters have different roles. The parameter Γ quantifies two-photon absorption relative to the Kerr nonlinearity and can be changed physically through the material response and operating wavelength; σ and λ are determined by n_4 and n_6 together with the wavelength and core radius and characterize the fifth- and seventh-order conservative nonlinearities. These normalized parameters are small relative to the cubic coefficient, which is scaled to unity. In the calculations below, $\sigma=0.01$ and $\lambda=0.001$ are kept fixed. Accordingly, the normalized Schrödinger equation (5) can be written as

$$i \frac{\partial \Psi}{\partial z} = -\frac{1}{2} \nabla^2 \Psi + V(r) \Psi - (1 + i \Gamma) |\Psi|^2 \Psi + \sigma |\Psi|^4 \Psi + \lambda |\Psi|^6 \Psi + i \gamma \cdot \Psi \quad (12)$$

with the linear potential reads

$$V(r) = -k^2 \tilde{r}_c^2 \delta n(r). \quad (13)$$

For the SI fiber, this potential is expressed as

$$V(r) = \begin{cases} V_{core} = -k^2 \tilde{r}_c^2 (n_{core} - n_{clad}), & 0 \leq r < 1 \\ V_{clad} = 0, & r \geq 1 \end{cases} \quad (14)$$

The linear potential of the GRIN fiber is given by

$$V(r) = \begin{cases} V_{core} = -k^2 \tilde{r}_c^2 n_{core} (1 - r^2), & 0 \leq r < 1 \\ V_{clad} = 0, & r \geq 1 \end{cases} \quad (15)$$

To solve the nonlinear propagation equation (12) with the potentials (14) and (15), numerical methods must be used, because analytical techniques are not effective for this higher-order nonlinear partial differential equation, which includes additional terms describing beam absorption and gain [28]-[31]. Finding stable propagating states of optical beams in this system is highly challenging. It is important to note that, because the model contains absorption and gain terms, the optical fiber under consideration is a non-Hermitian system. Standard numerical methods for conservative, or Hermitian, systems, such as the accelerated imaginary-time propagation method (AITM), the squared-operator method (SOM), and the Newton-conjugate-gradient (NCG) method [28]-[31], cannot be applied directly to the present system. To search for dissipative soliton beams in this non-Hermitian fiber, we implement an integrated numerical

procedure consisting of several steps to obtain stable beams. The computational procedure for finding dissipative spatial solitons is as follows.

In the first step, the stationary solutions of Eq. (12) in the conservative case, corresponding to $\gamma = \Gamma = 0$, are computed using standard numerical algorithms that are effective for nonlinear systems [30]-[31]. For fundamental soliton beams, the accelerated imaginary-time propagation method is used. For topological solitons, or vortices, the SOM and NCG algorithms are applied. The input beam launched into the fibers in the first step of the computation is chosen as a Gauss-Laguerre beam of the form

$$\Psi(\mathbf{r}) = \Psi_0 \sqrt{\frac{2}{\pi|m|!}} \left(\frac{\sqrt{2}}{\omega} r \right)^{|m|} \exp\left(-\frac{r^2}{\omega^2} + im\varphi\right), \quad (16)$$

where beams with $m = 0$ are used to search for fundamental solitons, whereas beams with $m=1$ are used to search for vortices.

After obtaining the spatial solitons of the conservative systems, we perform the second step by adiabatically introducing the nonconservative parameters in Eqs. (8)-(9) into Eq. (12) through a real-space propagation simulation of the beams. The method used in this stage is the second-order split-step Fourier algorithm (SSFM) [1].

In the third step, once the nonconservative parameters have reached their target values, their variation is stopped. The beam is then propagated further in real space with fixed nonconservative parameters. After a sufficiently long propagation distance, if the beam profile and phase distribution remain unchanged, the beam is considered stable, and the desired robust dissipative soliton beam is obtained.

The individual algorithms used in the three stages are established methods. Step 1 uses AITM/SOM/NCG only in the conservative limit to generate stationary seed states, whereas Steps 2 and 3 use the SSFM to propagate the full non-Hermitian equation, first while γ and Γ are introduced adiabatically and then with these parameters fixed. The methodological contribution is this continuation-and-validation workflow, which avoids applying conservative stationary solvers directly to the dissipative target equation (12).

3. Results and discussion

In this section, we present results for the search for stable dissipative spatial soliton beams using the computational and simulation procedure described above. The results demonstrate the existence of stable fundamental and vortex beams in the non-Hermitian SI and GRIN optical fibers considered here, although the parameter domains in which these beams remain stable are relatively narrow. The results are shown in Figures. 2-5.

It should be noted that stationary states of beams propagating in the optical fiber can be written in the form

$$\Psi(\mathbf{r}, z) = \Phi(\mathbf{r}) \cdot \exp(i\beta z). \quad (17)$$

The characteristics of these beams include the power P , the propagation constant β , the mean beam width $\langle W \rangle$ along the radial direction of the fiber, and the spatial phase distribution. The beam power is defined by the integral expression

$$P = \int |\Psi(\mathbf{r}, z)|^2 d\mathbf{r} = \int |\Phi(\mathbf{r})|^2 d\mathbf{r}. \quad (18)$$

By substituting this expression into the propagation equation, we obtain the propagation constant as

$$\beta = -\frac{1}{P} \int \left[-\frac{1}{2} |\nabla \Phi|^2 + (V(r) + i\gamma) |\Phi|^2 - (1 + i \cdot \Gamma) |\Phi|^4 + \sigma \cdot |\Phi|^6 + \lambda \cdot |\Phi|^8 \right] d\mathbf{r}. \quad (19)$$

The mean beam width is defined as

$$\langle W \rangle = \frac{\sqrt{\pi}}{P} \int |\Psi(\mathbf{r}, z)|^2 |\mathbf{r}| d\mathbf{r}. \quad (20)$$

To determine whether a beam is stable, we monitor the evolution of the mean beam width $\langle W \rangle$, the maximum field amplitude $\max(|\Psi|)$, and the beam power during the propagation simulations.

The non-Hermitian character of the fibers can be seen directly from the power balance. From Eqs. (12) and (18), $dP/dz = 2 \gamma P - 2 \Gamma \int |\Psi|^4 d\mathbf{r}$, which is not zero in general. Therefore, linear gain and two-photon absorption render the propagation generator non-self-adjoint, and the optical power is not conserved; the conservative Hermitian limit is recovered when $\gamma = \Gamma = 0$.

Numerical convergence is checked by evaluating the absolute error between two successive iterations, specifically

$$\text{Absolute error}(n) = \left| \left| \Psi \right|_{\max}^{(n+1)} - \left| \Psi \right|_{\max}^{(n)} \right|. \quad (21)$$

A convergence error below 10^{-4} is considered acceptable. The stability of the solutions is tested by adding small-amplitude noise and performing direct propagation using the SSFM algorithm.

To complement the stationary-error criterion, the converged fields are perturbed by small random noise and propagated over a long distance with the full SSFM dynamics while the mean width, maximum field amplitude, beam power, and transverse profiles are monitored. A full spectral eigenvalue analysis would require solving a large non-self-adjoint two-dimensional linearized eigenproblem for the higher-order gain-loss model and is beyond the scope of the present study; the noisy direct-propagation test is therefore used here as an additional dynamical stability indicator.

The parameter sets in Figures 2-5 were selected from preliminary scans of the normalized model to provide representative localized states and to identify the ranges in which stationary propagation is obtained. To make the SI/GRIN comparison consistent, $V_{\text{core}}=2.5$, $\sigma=0.01$, and $\lambda=0.001$ are fixed. For the stable families, $\Gamma=0.05$ is used, and the pump parameter is scanned over $\gamma=0.005$ -0.0275 (SI-fundamental), $\gamma=0.005$ -0.045 (GRIN-fundamental), and $\gamma=0.005$ -0.035 (GRIN-vortex); the unstable SI-vortex example uses $\gamma=0.005$ and $\Gamma=0.002$. The mean width, maximum field amplitude, and power plotted during propagation are not independently selected parameters but diagnostic quantities determined by the evolving solution. The input power is varied only to construct the $\beta(P)$ branches, while γ is varied to determine the pump range supporting stable states.

We first consider SI fibers. For this type of optical fiber, numerical calculations using the input function (16) with $m=0$ show that a fundamental soliton beam of Eq. (12) can be found and remains highly stable during propagation. Figure 2(a) illustrates

the calculated evolution of the characteristic beam quantities as functions of the propagation distance z in the fiber, evaluated using Eqs. (18)-(20). For $z > 300$, the parameters γ and Γ are held fixed. At this stage, the characteristic quantities of the beam become steady and remain nearly unchanged, confirming the stability of the beam. Figure 2(b) shows the field distribution of the beam in the plane transverse to the fiber axis after stable propagation has been reached. To evaluate the accuracy of this numerical procedure, we calculate the error defined in Eq. (21) as the propagation distance z increases, while keeping the parameters γ and Γ fixed. The evolution of the error is shown in Figure 2(c). It is evident that the error decreases to 10^{-5} as the numerical procedure converges. To assess the stability of the final state obtained from this calculation, we introduce a small perturbation to the solution and perform a direct propagation simulation. Figure 2(d) shows the cross sections of the soliton solution along the radial direction of the fiber during the propagation process. The soliton profile remains essentially unchanged, indicating that the obtained soliton is stable.

Next, we vary the power of the initial beam launched into the optical fiber in the numerical procedure described above. Figure 2(e) shows the dependence of the propagation constant β on the beam power P after stable propagation is reached. The numerical results indicate that the dependence is linear in the low-power regime. As P increases, the growth rate of β decreases, revealing a nonlinear dependence. Finally, we vary the power of the external pump applied along the fiber axis, corresponding to a change in the gain parameter γ , and determine the corresponding beam powers that can sustain stable propagation. These results are illustrated in Figure 2(f).

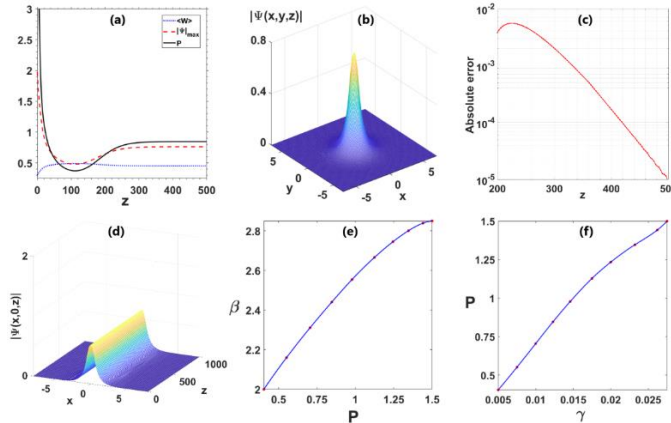


Figure 2. Formation of fundamental soliton beams in a higher-order non-Hermitian nonlinear optical fiber with a step-index profile, $V_{core} = 2.5$, $\gamma = 0.005 \div 0.0275$, $\Gamma = 0.05$, $\sigma = 0.01$, $\lambda = 0.001$. Panels (b) and (d) show an example of a stable fundamental beam profile and its cross section along the radial direction during propagation for the particular parameter set $\gamma = 0.0125$, $\beta = 2.41$, $P = 0.84$.

For beams with a topological structure, namely optical vortices, we use the input beam given by Eq. (16) with $m=1$. Through systematic calculations over a wide range of physical parameters in Eq. (12), we conclude that vortices cannot propagate stably in SI fibers. Figures 3(a) and 3(b) show the evolution of a vortex in the SI fiber. The initially unstable vortex beam, whose profile is shown in Figure 3(c), evolves into the fundamental beam shown in Figure 3(d) during propagation.

Next, we examine the stability of beams propagating in GRIN fibers. Figure 4(a) describes the evolution of the characteristic parameters of a fundamental beam during the computation for a representative parameter set. The propagation stage with $z > 250$ shows that the parameters have reached steady values, while the parameters γ and Γ are held fixed. This indicates that the beam propagates stably in the GRIN fiber. Figure 4(b) shows the field distribution of the beam in the transverse plane. The error defined by Eq. (21) decreases below the tolerance of 10^{-4} , as shown in Figure 4(c). Figure 4(d) shows the cross sections of the fundamental soliton beam along the radial direction of the fiber during its propagation. Figure 4(e) shows the dependence of the beam propagation constant β on its power P . The dependence is linear in the low-power regime and then increases more rapidly as P increases, indicating a nonlinear response different from that observed for the SI fiber in Figure 2(e). As the external pump power increases, γ rises, and the stable beam power also increases. The numerical results shown in Figure 4(f) are similar to those in Figure 2(f).

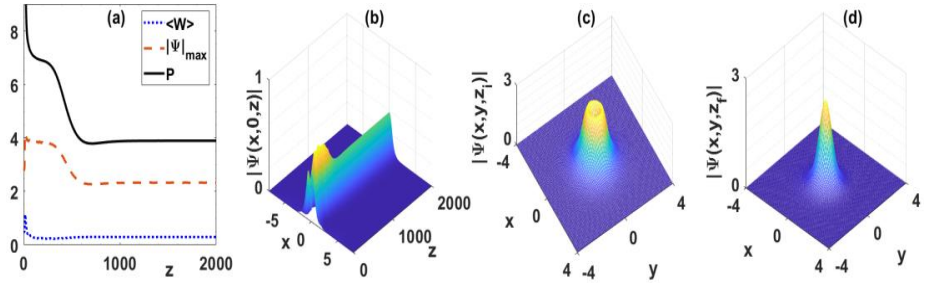


Figure 3. An unstable vortex soliton beam in a higher-order non-Hermitian nonlinear optical fiber with a step-index profile, with $V_{\text{core}} = 2.5$, $\gamma = 0.005$, $\Gamma = 0.002$, $\sigma = 0.01$, $\lambda = 0.001$. The initial vortex turns into the fundamental soliton during propagation. Panels (a) and (b) depict the evolution of the vortex soliton; panel (c) shows the initial vortex found using the SOM with $P = 18$, and panel (d) shows the final fundamental soliton with $P = 3.8$.

When the input beam in the numerical procedure is a vortex, the results show that vortices can propagate stably in the GRIN fiber. Figure 5(a) shows the evolution of the characteristic parameters of a vortex beam during the computation for the parameter values in Eq. (12), $\sigma = 0.01$, $\lambda = 0.001$, where the gain and absorption parameters are slowly increased to $\gamma = 0.0125$ and $\Gamma = 0.05$, respectively. After a propagation distance of $z > 300$, the vortex reaches a stationary state and then remains unchanged. The corresponding propagation constant and power of the vortex beam are determined to be $\beta = 4.42$ and $P = 2.22$, respectively. Figure 5(b) shows the field distribution of the vortex in the plane transverse to the propagation direction. The characteristic annular structure with a central intensity dip, typical of a vortex beam, can be observed. Figure 5(e) illustrates the dependence of the propagation constant β on the vortex-beam power P . A linear dependence is clearly observed in the low-power regime, followed by a nonlinear dependence as P increases, similar to the case of the fundamental beam in Figure 4(e).

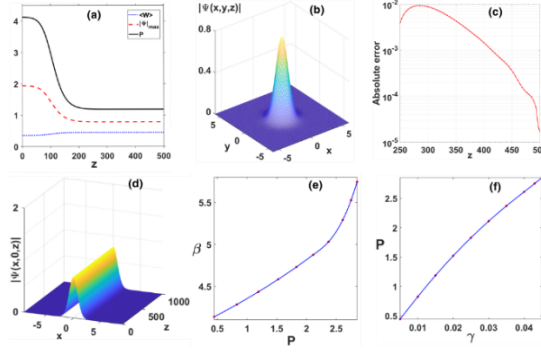


Figure 4. Formation of fundamental soliton beams in a higher-order non-Hermitian nonlinear optical fiber with a graded-index profile, with $V_{core} = 2.5$, $\gamma = 0.005 \div 0.045$, $\Gamma = 0.05$, $\sigma = 0.01$, $\lambda = 0.001$. Panels (b) and (d) show an example of a stable fundamental beam and its cross section during propagation for the parameter set $\gamma = 0.015$, $\beta = 4.43$, $P = 1.19$.

Finally, we investigate how the power P of stable vortex beams propagating in the GRIN fiber depends on the external pump power. Figure 5(f) shows that P increases almost linearly with the gain parameter γ of the pump source.

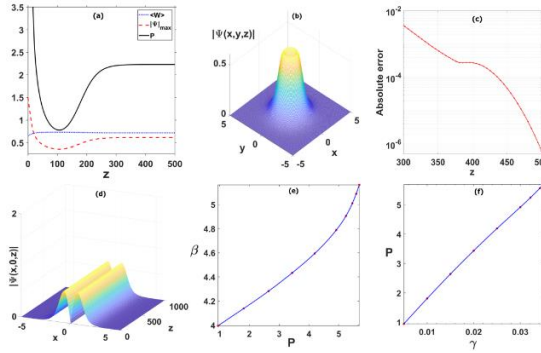


Figure 5. Formation of topological soliton beams (vortices) in a higher-order non-Hermitian nonlinear optical fiber with a graded-index profile, with $\gamma = 0.005 \div 0.035$, $\Gamma = 0.05$, $\sigma = 0.01$, $\lambda = 0.001$. Panels (b) and (d) show an example of a stable vortex beam and its cross section during propagation for the parameter set $\gamma = 0.0125$, $\beta = 4.42$, $P = 2.22$.

The contrasting vortex behavior in the two fiber types can be understood from the linear guiding potentials. Inside the SI core the potential is essentially flat, and the radial restoring action is concentrated at the abrupt core boundary. An annular vortex therefore has relatively weak distributed radial confinement, so perturbations in the presence of gain, nonlinear loss, and nonlinear refraction can drive radial and azimuthal redistribution, and the phase singularity may be lost, as observed in Figure 3. In the GRIN fiber, the smooth, approximately parabolic index profile produces a distributed radial restoring potential throughout the core. This continuous confinement limits radial excursions and helps preserve the annular intensity distribution and phase circulation, allowing diffraction, nonlinear refraction, gain, and absorption to reach a stationary balance. This provides a physical interpretation of why the vortex family is stabilized in the GRIN case but not in the SI case within the parameter ranges explored here.

4. Conclusions

In this paper, we have presented a study of the existence and stability of fundamental soliton beams and vortex beams propagating in higher-order nonlinear optical fibers with two-photon absorption and external energy pumping. By constructing an integrated numerical procedure with algorithms suited to each computational stage, we successfully identify nonlinear modes in these complex systems and examine their characteristics as the physical parameters vary. The numerical results show that, for SI fibers, only stable fundamental modes exist, whereas robust vortices cannot be formed. Our calculations also demonstrate that both fundamental beams and vortex beams can remain stable in GRIN fibers. The dependence of vortex-beam stability on the refractive-index profile of non-Hermitian fibers represents a novel theoretical prediction of the present work. The individual numerical algorithms are established methods; the methodological contribution lies in their coordinated three-stage continuation-and-validation workflow for the non-Hermitian SI/GRIN problem. Experimental validation of this profile-dependent vortex stability remains an important direction for future work.

Note for contributor:

- Short bio: Nguyen Viet Hung is a lecturer at the School of Materials Science and Engineering, Hanoi University of Science and Technology, Vietnam.

- Author's contributions: Nguyen Viet Hung: conceptualization, methodology, numerical simulation, data analysis, writing, review & editing.

Conflict of interest. The author declares no conflict of interest.

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