

SPECTRAL RADIUS CONDITIONS FOR CONNECTED PLANAR GRAPHS TO HAVE A FEW CUT-EDGES

Bach Viet Hoang

University of Transport Technology, Hanoi, Vietnam

*Corresponding author: Bach Viet Hoang, e-mail: bachviethoang01@gmail.com

Received June 1, 2026. Revised July 7, 2026. Accepted September 30, 2026.

Abstract. Let G be a graph of order n and let $\rho(G)$ be the spectral radius of the graph G . A cut-edge of a connected graph is an edge whose deletion increases the number of components. In this paper, we first state a condition for a planar graph to have a few cut-edges. After that, we give a sufficient condition on the spectral radius for a planar graph to have a few cut-edges.

Keywords: planar graph, spectral radius, cut-edge.

1. Introduction

In this paper, we only consider undirected and simple graphs. Let G be a graph with vertex set $V(G) = \{v_1, v_2, \dots, v_n\}$ and edge set $E(G)$. If any pair of vertices v_i and v_j are adjacent, then we write $v_i v_j \in E(G)$. We denote the number of elements in X by $|X|$. Let $m = |E(G)|$. A cut-edge of a connected graph is an edge whose deletion increases the number of components. If G is a connected graph of order n , then it is well known that G contains at most $n - 1$ cut-edges. The connected graphs of order n with $n - 1$ cut-edges must be trees. However, if the additional constraints of a connected graph are given, then the problem of determining the maximum number of cut-edges is nontrivial. Many studies on this topic have been stated (see [1]-[4] for examples).

On the other hand, the adjacency matrix of G , denoted by $A(G)$, is defined by its entries a_{ij} are equal to 1 if $v_i v_j \in E(G)$ and 0 otherwise. Let $\rho_1(G) \geq \rho_2(G) \geq \dots \geq \rho_n(G)$ be the eigenvalues of $A(G)$. We say $\rho(G) := \rho_1(G)$ is the spectral radius of G . The spectral radius of a graph contains lots of information about the graph. Many results on the spectral radius of a graph have been reported in the existing literature (see [5]-[9] for examples).

Motivated by the above results, in this paper we study the spectral conditions for a graph to have a few cut-edges.

To achieve the main results, we first study the maximum number of cut-edges under condition on the size of a connected planar graph. After that, we give a spectral radius condition for a graph to have a few cut-edges.

2. Main Results

Definition 2.1. [10] Let H be a connected graph with exactly $k + 1$ cut-edges $\{e_i\}, i \in \{1, 2, \dots, k + 1\}$ and $H - \{e_i | i \in \{1, 2, \dots, k + 1\}\}$ has $k + 2$ components $C_j, j \in \{1, 2, \dots, k + 1\}$ such that $|V(C_i)| = 1$ for all $i \in \{1, 2, \dots, k + 1\}$ and C_{k+2} is a connected planar graph with maximum edges. Let $\mathcal{H}$ be the set of such graphs H .

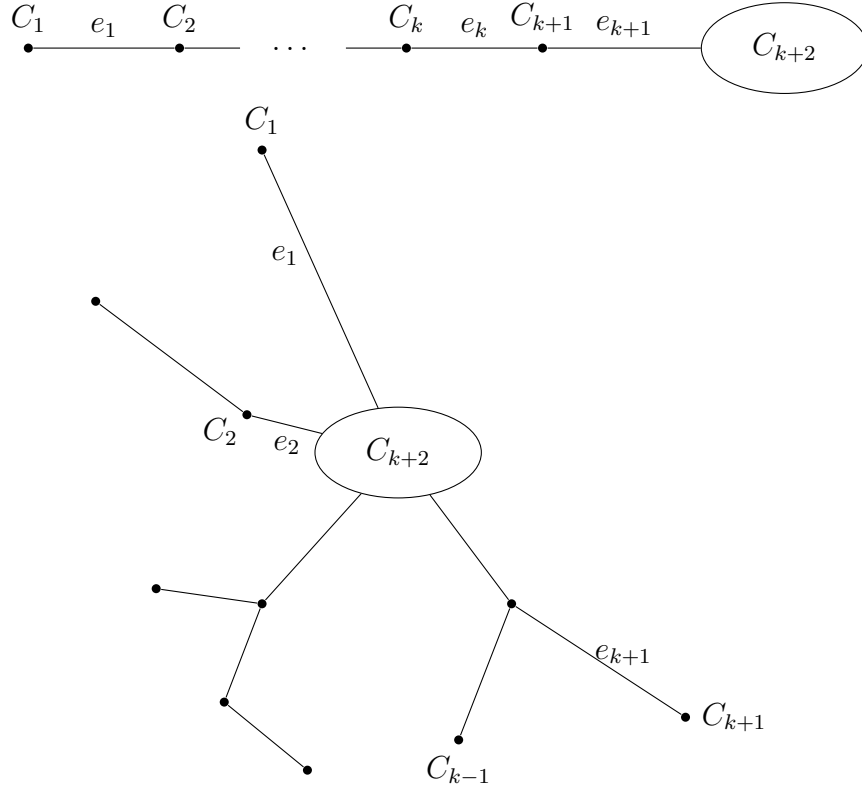


Figure 1. Two examples of H in $\mathcal{H}$

Recently, Ha, Hanh and Minh [10] proved the following.

Theorem 2.1. [10] Consider two positive integers n, k . Let G be a connected planar graph of order n and $m = |E(G)|$. If $m > 3n - 2k - 9$, then G has at most k cut-edges unless one of the following cases holds:

- i) G is isomorphic to a graph H in $\mathcal{H}$.
- ii) G is isomorphic to a tree of order $k + 2$.
- iii) G is isomorphic to a tree of order $k + 3$.

The first main purpose of this paper is to improve Theorem 2.1 by replacing the condition $m > 3n - 2k - 9$ by $m \geq 3n - 2k - 9$. To achieve this result, we find all graphs of order n with size $3n - 2k - 9$ to have at least $k + 1$ cut-edges. We now introduce the following definition.

Definition 2.2. Let H' be a connected graph with exactly $k + 1$ cut-edges $\{e_i\}, i \in \{1, 2, \dots, k + 1\}$ and $H - \{e_i | i \in \{1, 2, \dots, k + 1\}\}$ has $k + 2$ components $C_j, j \in \{1, 2, \dots, k + 2\}$ such that $|V(C_i)| = 1$ for all $i \in \{1, 2, \dots, k + 1\}$ and C_{k+2} is a connected planar graph whose number of edges equals the maximum number of edges minus 1. Define $\mathcal{H}'$ as the set of such graphs H' .

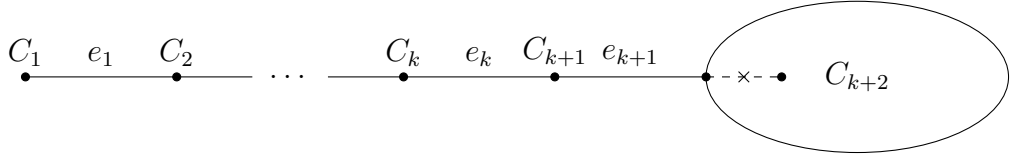


Figure 2. An example of H' in $\mathcal{H}'$

We now state the following.

Theorem 2.2. Consider two positive integers n, k . Let G be a connected planar graph of order n and $m = |E(G)|$. If $m > 3n - 2k - 10$, then G has at most k cut-edges unless one of the following cases holds:

- i) G is isomorphic to a graph H in $\mathcal{H}$.
- ii) G is isomorphic to a graph H' in $\mathcal{H}'$.
- iii) G is isomorphic to a tree of order $k + 2$.
- iv) G is isomorphic to a tree of order $k + 3$.
- v) G is isomorphic to a tree of order $k + 4$.

Proof. If $m > 3n - 2k - 9$, then the proof of this Theorem is complete by Theorem 2.1. We consider the situation where $m = 3n - 2k - 9$.

Assume for the sake of contradiction that G has strictly more than k cut-edges. Let $\{e_1, e_2, \dots, e_{k+1+r}\}$ (with $r \in \mathbb{N}$) denote the set of all cut-edges of G and $G \setminus \{e_1, e_2, \dots, e_{k+1+r}\} = \{C_1, C_2, \dots, C_{k+2+r}\}$ be the set of connected components obtained after removing the cut-edges. Set $n_i = |V(C_i)|$ for every $i \in \{1, 2, \dots, k+2+r\}$. Here, we note that we may assert $C_i \neq K_2$ for all $i \in \{1, 2, \dots, k+2+r\}$. Indeed, if there exists some $C_i = u_i v_i$, then the edge $u_i v_i$ is considered as a cut-edge of G . Thus, we obtain that $n_i = 1$ or $n_i \geq 3$ for every $i \in \{1, 2, \dots, k+2+r\}$. Since G is planar, each component C_i is also planar. Therefore, if $n_i \geq 3$, we have $|E(C_i)| \leq 3n_i - 6$.

We consider the following three cases:

Case 1: $n_i \geq 3$ for all $i \in \{1, 2, \dots, k+2\}$. We conclude that

$$\begin{aligned} m &= \sum_{i=1}^{k+2+r} |E(C_i)| + (k+1+r) \leq \sum_{i=1}^{k+2+r} (3n_i - 6) + k+1+r \\ &= 3 \sum_{i=1}^{k+2+r} n_i - 6(k+2+r) + k+1+r = 3n - 5k - 11 - 5r \leq 3n - 2k - 11, \end{aligned}$$

this gives a contradiction.

Case 2: There exist some $i, j \in \{1, 2, \dots, k+2\}$ such that $n_i = 1$ and $n_j \geq 3$. Without loss of generality, we may assume that $n_1 = n_2 = \dots = n_p = 1$ and $n_j \geq 3$ for all $j \in \{p+1, p+2, \dots, k+2+r\}$. Hence, we have $|E(C_i)| = 0, \forall i \in \{1, 2, \dots, p\}$ and $|E(C_j)| \leq 3n_j - 6, \forall j \in \{p+1, p+2, \dots, k+2+r\}$. Therefore, we obtain that

$$\begin{aligned} m &= \sum_{i=1}^p |E(C_i)| + \sum_{j=p+1}^{k+2+r} |E(C_j)| + (k+1+r) \leq 0 + \sum_{j=p+1}^{k+2+r} (3n_j - 6) + k+1+r \\ &= 3(n-p) - 6(k+2+r-p) + k+1+r = 3n - 5k - 11 + 3p - 5r. \end{aligned} \tag{2.1}$$

On the other hand, by definition we have $p \leq k+1+r$ and $r \geq 0$. Then, we conclude that

$$\begin{aligned} m &\leq 3n - 5k - 11 + 3p - 5r \leq 3n - 5k - 11 + 3(k+1+r) - 5r \\ &= 3n - 2k - 8 - 2r. \end{aligned}$$

If $r \geq 1$ then $m \leq 3n - 2k - 10$, contradicting $m = 3n - 2k - 9$.

If $r = 0$ then from formula (2.1) we have $m \leq 3n - 5k - 11 + 3p$. Combining with $|E(G)| = 3n - 2k - 9$, we get

$$3n - 2k - 9 \leq 3n - 5k - 11 + 3p,$$

which implies $p \geq k + \frac{2}{3}$. Hence $p = k+1$. Then

$$|E(C_{k+2})| = m - (k+1) = 3n - 2k - 9 - (k+1) = 3(n-k-1) - 7 = 3n_{k+2} - 7,$$

This implies that G is isomorphic to a graph H' in $\mathcal{H}'$.

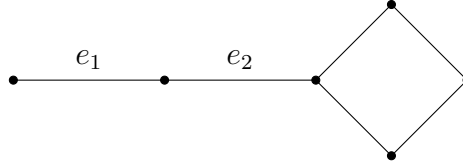
Case 3: $n_i = 1$ for all $i \in \{1, 2, \dots, k+2+r\}$. Thus, $n = k+2+r$. Hence, we have

$$m = k+1+r = 3(k+2+r) - 2k - 5 - 2r = 3n - 2k - 5 - 2r.$$

Combining this with $m = 3n - 2k - 9$ we obtain $r = 2$. Thus, G is a tree with $k+3$ cut-edges. This means that G is a tree of order $k+4$.

The proof is completed. □

We now give an example to show that the condition of Theorem 2.2 is sharp. Let G be a graph of order $n = 6$ as depicted below.



Let $k = 1$. Then G has $k + 1$ cut-edges e_1, e_2 and $|E(G)| = 3n - 2k - 10 = 6$. This shows that the result is sharp.

Theorem 2.2 allows us to deduce the following result on the spectral radius

Theorem 2.3. *Let n, k be positive integers. Let G be a connected planar graph of order n . If*

$$\rho(G) > \sqrt{5n - 4k - 19},$$

then G has at most k cut-edges unless one of the following cases holds:

- a) G is isomorphic to a graph H in $\mathcal{H}$.*
- b) G is isomorphic to a graph H' in $\mathcal{H}'$.*
- c) G is isomorphic to a tree of order $k + 2$.*
- d) G is isomorphic to a tree of order $k + 3$.*
- e) G is isomorphic to a tree of order $k + 4$.*

Proof. To prove Theorem 2.3, we use the following result of Hong [6].

Theorem 2.4. [6] *Let G be a graph with n vertices and m edges. Then,*

$$\rho(G) \leq \sqrt{2m - n + 1}.$$

The equality holds if and only if G is the star $K_{1,n-1}$ or the complete graph K_n .

By the assumption $\rho(G) > \sqrt{5n - 4k - 19}$ and Theorem 2.4, we have

$$5n - 4k - 19 < 2m - n + 1.$$

This holds if and only if

$$m > 3n - 2k - 10.$$

Now, according to Theorem 2.2, the proof of Theorem 2.3 is completed. $\square$

The condition of Theorem 2.2 is sharp, hence the threshold of Theorem 2.3 is also sharp.

3. Conclusions

We have extended existing results on the number of cut-edges in connected planar graphs. By meticulously analyzing the case where $m = 3n - 2k - 9$, we have identified additional exceptional graph structures, namely the family $\mathcal{H}'$ and trees of order $k+4$. This allowed us to reduce the edge-count requirement to $m > 3n - 2k - 10$ while maintaining a characterization of all exceptions. Furthermore, by applying Hong's bound [6], we translated these edge-density conditions into a spectral radius threshold. Theorem 2.3 provides a tight spectral condition for ensuring a planar graph has few cut-edges.

Note for contributor.

- Short bio: Bach Viet Hoang is a Lecturer at University of Transport Technology, Vietnam;
- Author's contributions: Bach Viet Hoang: Conceptualization, methodology, data analysis, writing & editing.

Conflict of interest. The authors declare no conflict of interest.

Acknowledgments. The author would like to thank to Assoc. Prof. Pham Hoang Ha for introducing the main research problem, providing crucial literature references, and offering initial directions for the manuscript.

REFERENCES

- [1] N. Achuthan, A. R. Rao, "On the number of cut edges in a regular graph", *Australasian Journal of Combinatorics*, vol. 27, pp. 5–12, 2003.
- [2] A. Ramachandra Rao, "An extremal problem in graph theory", *Israel Journal of Mathematics*, vol. 6, pp. 261–266, 1968.
- [3] A. Ramachandra Rao, "Some extremal problems and characterizations in the theory of graphs", Ph.D thesis, Indian Statistical Institute, 1969.
- [4] O. Suil and D.B. West, "Balloons, cut edges, matchings, and total domination in regular graphs of odd degree", *Journal of Graph Theory*, vol. 64, pp. 116–131, 2010.
- [5] J.-M. Guo, Z.-W. Wang and X. Li, "Sharp upper bounds of the spectral radius of a graph", *Discrete Mathematics* vol. 342 (2019), 2559–2563.
- [6] Y. Hong, "A bound on the spectral radius of graphs", *Linear Algebra and Its Applications*, vol. 108, pp. 135–139, 1988.
- [7] Y. Hong, J.-L. Shu and K. Fang, "A sharp upper bound of the spectral radius of graphs", *Journal of Combinatorial Theory, Series B*, vol. 81, no. 2, pp. 177–183, 2001.
- [8] V. Nikiforov, "Some inequalities for the largest eigenvalue of a graph", *Combinatorics, Probability and Computing*, vol. 11, no. 2, pp. 179–189, 2002.
- [9] S. Sun, K. C. Das. "A conjecture on the spectral radius of graphs". *Linear Algebra and Its Applications*, vol. 588, pp. 74-80, 2020.
- [10] P. H. Ha, D. D. Hanh, V. Q. Minh, "The number of cut-edges and conflict-free connection number in planar graphs", arXiv:2606.00533, 2026.