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APPROXIMATELY COHEN–MACAULAY PROPERTY OF EDGE-WEIGHTED CYCLES

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Abstract. In this paper, we investigate the approximately Cohen-Macaulay property of edge ideals associated with edge-weighted graphs. Our main result establishes that the edge ideal $I(C_n, \mathbf{w})$ of an edge-weighted cycle graph of order n is approximately Cohen-Macaulay if and only if $n = 3$ or $n = 5$. The proof heavily relies on the algebraic properties of sequentially Cohen-Macaulay modules, Woodroofe graphs, and Goto's characterization of approximately Cohen-Macaulay rings.
Keywords: approximately Cohen–Macaulay, edge-weighted graph, Woodroofe graph.

1. Introduction

Let $S = k[x_1, \dots, x_n]$ be a standard graded polynomial ring over an arbitrary field k. Let G be a simple graph with vertex set $V = \{x_1, \dots, x_n\}$ and edge set $E(G)$. By abuse of notation, we also use $x_i x_j$ to denote an edge $\{x_i, x_j\}$ of G. Assume that $\mathbf{w} : E(G) \to \mathbb{Z}_{>0}$ is a weight function on the edges of G. The edge ideal of the edge-weighted graph $(G, \mathbf{w})$ is defined by

$$I(G, \mathbf{w}) = \big((x_i x_j)^{\mathbf{w}(x_i x_j)} \mid x_i x_j \in E(G)\big) \subseteq S.$$

Before proceeding, we recall some basic notions from graph theory. A graph G consists of a finite vertex set $V(G) = \{x_1, \dots, x_n\}$ and an edge set $E(G)$. For each vertex $x_i \in V(G)$, the degree of x_i, denoted by $\deg(x_i)$, is the number of edges incident to it.

A directed graph (or digraph) D is a pair $(V(D), A(D))$, where $A(D)$ is a set of ordered pairs of vertices called arcs. For a vertex x_i in D, the out-degree, denoted by $\deg^+(x_i)$, is the number of arcs starting at x_i. Throughout this paper, we assume all graphs are simple, meaning they have no loops or multiple edges. In particular, if every edge of G has weight one then $I(G, \mathbf{w})$ becomes the usual edge ideal $I(G)$.

In [1], Paulsen and Sather-Wagstaff introduced edge ideals of edge-weighted graphs. A graph G (resp. $(G, \mathbf{w})$) is said to be Cohen–Macaulay if $I(G)$ (resp. $I(G, \mathbf{w})$) is. In particular, they proved that $(G, \mathbf{w})$ is Cohen–Macaulay for all weight functions $\mathbf{w}$ when G is a complete graph. On the other hand, the concept of an approximately Cohen–Macaulay ring was introduced by Goto [2] as a different generalization of the Cohen–Macaulay ring, specifically for local rings. A graded version of this notion is also defined in [3].

In this paper, we extend these considerations by focusing on the approximately Cohen–Macaulay threshold. Our main result provides a complete characterization of weighted cycles $(C_n, \mathbf{w})$ that satisfy this property, establishing the conditions under which these graphs reach the approximately Cohen–Macaulay property.

Main Theorem. Let C_n be a cycle graph of order n and $\mathbf{w}$ be any weight function on the edges of C_n. The following conditions are equivalent:

(i) $I(C_n, \mathbf{w})$ is approximately Cohen-Macaulay;

(ii) $n = 3$ or $n = 5$.

2. Approximately Cohen–Macaulay weighted cycles

Throughout the paper, we denote $\mathfrak{m} = (x_1, \dots, x_n)$ the maximal homogeneous ideal of S. Let $\mathbf{w} : E(G) \to \mathbb{Z}_{>0}$ be a weight function on the edges of G. For an exponent $\underline{k} \in \mathbb{N}^n$, we denote by $\underline{x}^{\underline{k}}$ the monomial $x_1^{k_1} \dots x_n^{k_n}$ in S.

A simple graph G is called a Woodroofe graph if it has no chordless cycles of length other than 3 or 5.

Let $R = S/I$ be a standard graded k-algebra. R is said to be sequentially Cohen-Macaulay if there exists a finite filtration of homogeneous ideals $0 = I_0 \subset I_1 \subset \cdots \subset I_r = R$ such that each quotient I_j/I_{j-1} is Cohen-Macaulay and the dimensions of these quotients are strictly increasing:

$$\dim(I_1/I_0) < \dim(I_2/I_1) < \cdots < \dim(I_r/I_{r-1}).$$

The notion of sequentially Cohen-Macaulay was introduced by Stanley as a generalization of the Cohen–Macaulay property. It is well-known that a Woodroofe graph is vertex-decomposable, which implies it is sequentially Cohen–Macaulay. Furthermore, Woodroofe graphs are precisely the graphs for which the edge-weighted ideal $(G, \mathbf{w})$ is sequentially Cohen-Macaulay for all weight functions $\mathbf{w}$ [4].

As an analogue to Goto's notion of approximately Cohen–Macaulay rings, one can define a graded version of it.

Definition 2.1. *Let $R = \bigoplus_{i \geq 0} R_i$ denote a standard graded k-algebra with $R_0 = k$. It is called an approximately Cohen–Macaulay ring if there exists a homogeneous element x of positive degree such that $R/x^n R$ is Cohen–Macaulay of dimension* $\dim R - 1$ *for all* $n \geq 1$.

Let I be a homogeneous ideal in S and $R = S/I$ be a standard graded algebra. Suppose that $I = \bigcap_{i=1}^{s} \mathfrak{q}_i$ is a reduced primary decomposition of I and let $\mathfrak{p}_i$ denote the radical $\sqrt{\mathfrak{q}_i}$ of $\mathfrak{q}_i$ for $1 \leq i \leq s$. We define $U_S(I)$ as the equidimensional part of the primary decomposition of the ideal I, that is,

$$U_S(I) = \bigcap_{\dim(S/\mathfrak{p}_j)=\dim(S/I)} \mathfrak{q}_j.$$

There are several equivalent characterizations to determine when a graded ring is approximately Cohen–Macaulay. The following lemma provides a standard graded analogue of Goto's criteria [2], which was originally stated for Noetherian local rings.

Lemma 2.1. *Let I be a homogeneous ideal in S, and let* $\dim(S/I) = d$. *Then S/I is approximately Cohen–Macaulay if and only if $S/U_S(I)$ is a Cohen–Macaulay ring of dimension d and* $\operatorname{depth}(S/I) \geq d - 1$.

Proof. Let $R = S/I$ and $\mathfrak{m}$ be the unique maximal homogeneous ideal of S. Let $R_\mathfrak{m}$ denote the localization of R at $\mathfrak{m}$. According to the classical result of Goto [2], the assertion holds for the local ring $R_\mathfrak{m}$, namely, $R_\mathfrak{m}$ is approximately Cohen–Macaulay, if and only if $R_\mathfrak{m}/U(R_\mathfrak{m})$ is Cohen–Macaulay of dimension d and $\operatorname{depth}(R_\mathfrak{m}) \geq d - 1$.

To pass from the local case to the graded case, we observe that the localization functor behaves well with respect to the homogeneous structures. Specifically, since I is a homogeneous ideal, the primary components $\mathfrak{q}_j$ of I in its reduced primary decomposition are also homogeneous. Thus, the formation of the equidimensional part $U_S(I)$ commutes with localization at $\mathfrak{m}$, yielding $(U_S(I))_\mathfrak{m} = U_{S_\mathfrak{m}}(I_\mathfrak{m})$, which implies $(S/U_S(I))_\mathfrak{m} \cong R_\mathfrak{m}/U(R_\mathfrak{m})$.

Furthermore, it is a standard fact in graded ring theory that the Krull dimension, depth, and the property of being (approximately) Cohen–Macaulay of a standard graded algebra R over a field k are precisely equivalent to those of its local ring $R_\mathfrak{m}$ at the maximal homogeneous ideal. Therefore, the graded equivalence follows immediately from the local version. □

To avoid any potential confusion between the edges of the underlying undirected graph G and the arcs of a directed graph used in our proofs, we adopt the following convention: an edge in G connecting vertices x_i and x_j is denoted by the set $\{x_i, x_j\}$ (or simply $x_i x_j$ when referring to the corresponding monomial). In contrast, a directed arc from x_i to x_j in a digraph D is strictly denoted by the ordered pair $(x_i, x_j) \in A(D)$.

We first require the following auxiliary result.

Lemma 2.2. *Let G be a finite directed simple graph. If every vertex of G has an out-degree of at least 1, there always exists at least one directed cycle in G.*

Proof. Assume that G is a simple directed graph with finite vertex set $V = \{x_1, \dots, x_n\}$ and arc set A such that

$$\deg^+(x_i) \geq 1, \ \ \forall i = 1, \dots, n.$$

Let u_0 be a vertex in V, then there exists a vertex $u_1 \in V \setminus \{u_0\}$ such that $(u_0, u_1) \in A$. By repeating this process, we obtain a sequence of vertices $\{u_k\}$ where $u_{k+1} \in V \setminus \{u_k\}$ and $(u_k, u_{k+1}) \in A$ for all $k \geq 0$. Since the vertex set V is finite, by the Pigeonhole Principle, there must exist indices $i, j \in \mathbb{Z}_{>0}$ with $1 \leq i < j \leq n$ such that $u_i = u_j$. This implies the existence of a directed cycle $u_i, u_{i+1}, \dots, u_j$ passing through the vertex u_i in the graph. □

Proposition 2.1. *Let G be a simple graph. Then*

$$I(G, \mathbf{w}) : \mathfrak{m} = I(G, \mathbf{w}).$$

for all weight functions $\mathbf{w}$.

Proof. We just need to show that

$$I : \mathfrak{m} \subseteq I (\text{where } I = I(G, \mathbf{w})).$$

As $I : \mathfrak{m} = I : \left(\sum_{t=1}^{n} (x_t) \right) = \bigcap_{t=1}^{n} (I : (x_t)) = J$ is a monomial ideal in S, assume that $J \nsubseteq I$. Then there exists a monomial $f = \underline{x}^{\underline{k}} \in J$ such that $f \notin I$. Let $a_{ij} = \mathbf{w}(x_i x_j)$ for convenience.

- Since $f \notin I$, it follows that $k_i < a_{ij}$ or $k_j < a_{ij}, \ \ \forall x_i x_j \in E(G)$ (1).
- Since $f \in J$, we have $f \in I : (x_t), \ \ \forall t = 1, \dots, n$, which means $x_t f \in I, \ \ \forall t = 1, \dots, n$.

For all $t = 1, \dots, n$, let $x_t f = x_t . \underline{x}^{\underline{k}} \in I$. It follows that there exists an edge $x_i x_j \in E(G)$ such that $x_t \underline{x}^{\underline{k}} \in ((x_i x_j)^{a_{ij}})$. Assume that $t \notin \{i, j\}$, then $k_i, k_j \geq a_{ij}$, which contradicts (1). Thus, $x_t \underline{x}^{\underline{k}} \in I$ if and only if there exists an edge $x_t x_u \in E(G)$ such that $x_t \underline{x}^{\underline{k}} \in ((x_t x_u)^{a_{tu}})$.

We now define the arc set $A = \{(x_t, x_u) \mid x_t f \in ((x_t x_u)^{a_{tu}})\}$, since G is a simple graph with no multiple edges or self-loops, the condition $x_t \underline{x}^{\underline{k}} \in ((x_t x_u)^{a_{tu}})$ implies that $x_u \underline{x}^{\underline{k}} \notin ((x_t x_u)^{a_{tu}})$. Indeed, if both belong to the ideal, we should have $k_t + 1, k_u \geq a_{tu}$ and $k_t, k_u + 1 \geq a_{tu}$, which immediately contradicts (1). This ensures that the directed graph $G'(V, A)$ contains no symmetric pairs of arcs for the given monomial, thus is well-defined. These arguments imply the auxiliary directed graph $G'(V, A)$ has $\deg^+(x_t) \geq 1,\ \forall t = 1, \dots, n$.

By Lemma 2.2, the graph G' contains a directed cycle $C = (x_{i_1}, \dots, x_{i_s})$. Reorder the variables if necessary to assume that G' contains a directed cycle $C = (x_1, \dots, x_s)$ where $x_1, ..., x_s$ are distinct vertices. For each $i = 1, ..., s$, let a_i denote the weight associated with the edge $x_i x_{i+1}$ (with $x_{s+1} \equiv x_1$), which means $a_i = a_{i,i+1} = \mathbf{w}(i, i+1)$ in the reodered indexing. From $x_i.\underline{x}^{\underline{k}} \in ((x_i x_{i+1})^{a_i})$ $(i = 1, \dots, s)$, we show that $k_i = a_i - 1$ and $k_{i+1} \geq a_i$ by induction on i.

Base case: $i = 1$. Suppose that $x_1 \underline{x}^{\underline{k}} \in ((x_1 x_2)^{a_1})$, then $k_1 + 1 \geq a_1$ and $k_2 \geq a_1$. The condition (1) implies that $k_1 < a_1$. So we have $k_1 = a_1 - 1$ and $k_2 \geq a_1$, completing the base case.

Induction step: Assume that $i \geq 2$ and the result holds until $i - 1$. We prove that $k_i = a_i - 1$ and $k_{i+1} \geq a_i$. Indeed, the above argument implies that $x_i \underline{x}^{\underline{k}} \in ((x_i x_{i+1})^{a_i})$, which means $k_i + 1 \geq a_i$ and $k_{i+1} \geq a_i$. The condition (1) implies that $k_i < a_i$. So we have $k_i = a_i - 1$ and $k_{i+1} \geq a_i$, as desired.

Suppose that $k_{i+1} \geq a_i$ and $k_{i+1} = a_{i+1} - 1$, then $a_{i+1} - 1 \geq a_i$, which means $a_{i+1} > a_i$ for all $i = 1, \dots, s$. It follows that

$$a_1 = a_{s+1} > a_s > \cdots > a_2 > a_1.$$

which is impossible. □

The following corollary is a straightforward consequence of the above proposition.

Corollary 2.1. *Let G be a simple graph. Then* $\operatorname{depth}(S/I(G, \mathbf{w})) > 0$ *for all weight functions* $\mathbf{w}$.

Proof. Recall that $\operatorname{depth}(S/I(G, \mathbf{w})) = 0$ if and only if the maximal ideal $\mathfrak{m}$ is an associated prime of $S/I(G, \mathbf{w})$ (i.e., $\mathfrak{m} \in \operatorname{Ass}(S/I(G, \mathbf{w}))$). This happens if and only if there exists an element $f \in S \setminus I(G, \mathbf{w})$ such that $\mathfrak{m} f \subseteq I(G, \mathbf{w})$, which is equivalent to $I(G, \mathbf{w})\mathfrak{m} \neq I(G, \mathbf{w})$. Since Proposition 2.1 establishes $I(G, \mathbf{w}) : \mathfrak{m} = I(G, \mathbf{w})$, it follows that $\mathfrak{m} \notin \operatorname{Ass}(S/I(G, \mathbf{w}))$. Thus, $\mathfrak{m}$ contains a regular element on $S/I(G, \mathbf{w})$, which implies $\operatorname{depth}(S/I(G, \mathbf{w})) > 0$. □

Proof of main result

For one implication, assume that $I(C_n, \mathbf{w})$ is approximately Cohen–Macaulay. By Schenzel's result [5] on dimension filtration, every approximately Cohen–Macaulay graded ring is sequentially Cohen–Macaulay, which means $I(C_n, \mathbf{w})$ is sequentially Cohen–Macaulay. Thus G is a Woodroofe graph by [6] and the forward direction is straightforward from Woodroofe's result [4].

To see the converse, we consider the following cases.

Case 1: $n = 3$. Since C_3 is a complete graph K_3. The ideal $I(C_3, \mathbf{w})$ is Cohen–Macaulay due to [1], which in particular implies that it is approximately Cohen–Macaulay.

Case 2: $n = 5$. For the five-cycle C_5, which is a Woodroofe graph [4]. Hence, $I(C_5, \mathbf{w})$ is sequentially Cohen–Macaulay [6] and $S/U_S(I(C_5, \mathbf{w}))$ is Cohen–Macaulay follows from the definition of sequentially Cohen–Macaulay modules. By Corollary 2.1, $\operatorname{depth}(S/I(C_5, \mathbf{w})) \geq 1$ for all weight functions $\mathbf{w}$, which means $\operatorname{depth}(S/I(C_5, \mathbf{w})) \geq \dim(S/I(C_5, \mathbf{w})) - 1$ as $\dim(S/I(C_5, \mathbf{w})) = \dim(S/I(C_5)) = 2$. Hence, $S/I(C_5, \mathbf{w})$ is approximately Cohen–Macaulay by Lemma 2.2.

3. Examples and concluding remarks

In this section, we present a counterexample to show that the sequentially Cohen – Macaulay property of a weighted graph $(G, \mathbf{w})$ does not necessarily imply the approximately Cohen – Macaulay property. This observation highlights the non-trivial nature of the characterization provided in our main result.

We first recall some notations and results.

For a finitely generated graded S-module L, the depth of L is defined to be

$$\operatorname{depth} L = \min\{i \mid H^i_{\mathfrak{m}}(L) \neq 0\}$$

where $H^i_{\mathfrak{m}}(L)$ denotes the ith local cohomology module of L with respect to $\mathfrak{m}$. Hochster [7] proved that for a monomial ideal I, one has

$$\text{depth}(S/I) = \min\{\text{depth}(S/\sqrt{I : u}) \mid u \text{ is a monomial in } S, u \notin I\}. \tag{3.1}$$

An ideal of the form $\sqrt{I : u}$ is called an associated radical of I.

Lemma 3.1 ([6], Lemma 2.1)**.** *Let G be a simple graph and $\mathbf{w} : E(G) \to \mathbb{Z}_{>0}$ be a weight function. For any exponent $\underline{k} \in \mathbb{N}^n$, let*

$$U = \{i \mid \text{ there exists } j \text{ such that } \{x_i, x_j\} \in E(G) \text{ and } k_i < \mathbf{w}(x_i x_j) \leq k_j\}.$$

Then,

$$\sqrt{I(G, \mathbf{w}) : \underline{x}^{\underline{k}}} = I(G \setminus U) + (x_i \mid i \in U),$$

where $I(G \setminus U)$ is the edge ideal of the induced subgraph of G on $V(G) \setminus U$.

Example 3.1. *Let G be the star graph $K_{1,n}$ $(n \geq 3)$ with the vertex set $V(G) = \{x_0, x_1, x_2, ..., x_n\}$ and the edge set $E(G) = \{x_0x_1, x_0x_2, ..., x_0x_n\}$. Since G is a tree, it has no cycles, meaning it is trivially a Woodroofe graph. Therefore, its edge ideal $I(G, \mathbf{w})$ is sequentially Cohen – Macaulay for all weight functions $\mathbf{w}$. However, $I(G, \mathbf{w})$ is not necessarily approximately Cohen – Macaulay. Let us consider the trivial weight function $\mathbf{w} \equiv k \geq 1$. The edge weighted ideal is*

$$I(G, \mathbf{w}) = ((x_0x_1)^k, (x_0x_2)^k, ..., (x_0x_n)^k) \subset S = k[x_0, x_1, x_2, ..., x_n].$$

The primary decomposition of $I(G,\mathbf{w})$ *is* $(x_0^k) \cap (x_1^k, x_2^k, ..., x_n^k)$. *The minimal primes are* $\mathfrak{p}_1 = (x_0)$ *and* $\mathfrak{p}_2 = (x_1, x_2, ..., x_n)$. *Consequently, the Krull dimension of* $S/I(G)$ *is*

$$\dim(S/(I(G,\mathbf{w}))) = \max\{n+1-1, n+1-n\} = n.$$

The unmixed part is $U_S(I(G)) = (x_0^k)$. *Clearly,* $S/U_S(I(G))$ *is a Cohen – Macaulay ring of dimension* n *by assumption* $I(G,\mathbf{w})$ *is sequentially Cohen – Macaulay. On the other hand, pick* $\underline{k} = (k, k-1, ..., k-1)$. *It follows that* $U = \{x_1, x_2, ..., x_n\}$, *we have*

$$\begin{aligned}\operatorname{depth}(S/I(G,\mathbf{w})) &\leq \operatorname{depth}(S/(I(G\setminus U) + (x_i \mid i \in U)) \\ &\leq \dim(S/(I(G\setminus U) + (x_i \mid i \in U)).\end{aligned}$$

Since $S/(I(G\setminus U)+(x_i \mid i \in U)) = k[x_0, x_1, ..., x_n]/(x_1, ..., x_n)) \cong k[x_0]$, *it follows that*

$$\operatorname{depth}(S/I(G,\mathbf{w})) \leq \dim(S/(I(G\setminus U) + (x_i \mid i \in U)) = \dim k[x_0] = 1.$$

For the condition $I(G,\mathbf{w})$ *to be approximately Cohen – Macaulay, Lemma 2.2 requires:*

$$\operatorname{depth}(S/I(G,\mathbf{w})) \geq \dim(S/I(G,\mathbf{w})) - 1 = n - 1 \geq 2.$$

Since $1 \not\geq 2$, $I(G,\mathbf{w})$ *is not approximately Cohen – Macaulay.*

Remark 3.1. *As demonstrated in the example above, the approximately Cohen – Macaulay property is strictly stronger and much more restrictive than the sequentially Cohen – Macaulay property. While Woodroofe graphs completely characterize the sequentially Cohen – Macaulay property for weighted edge ideals, they do not guarantee the approximately Cohen-Macaulay condition. The presence of minimal primes with large height differences easily causes a drop in depth, violating the inequality* $\operatorname{depth}(S/I) \geq \dim(S/I) - 1$.

This highlights the complexity of the approximately Cohen – Macaulay property and justifies the necessity of the complete characterization for specific families, such as the cycle graphs presented in our Main Theorem. Furthermore, this observation naturally poses an open problem for future research: establishing a full combinatorial characterization of approximately Cohen – Macaulay edge ideals for broader classes of graphs, such as trees, bipartite graphs, or chordal graphs.

Note for contributor:

- Short bio: Phan Ha Son is a postgraduate student at the School of Mathematics and Computer Science, Hanoi National University of Education, Vietnam.

- Author's contributions: Phan Ha Son: conceptualization, methodology, software, data analysis, writing, visualization & editing.

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