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DYNAMIC EQUATIONS WITH TIME-VARYING DELAY ON TIME SCALES

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Abstract. This paper investigates dynamic delay equations on time scales. Due to the lack of a well-defined subtraction, the formulation of delay equations in this setting is nontrivial. We first introduce a suitable representation of delay dynamic equations on time scales. We then prove the existence and uniqueness of solutions, as well as the uniform exponential stability for Δ-dynamic delay equations via Lyapunov direct method. The obtained results extend existing works and provide verifiable conditions in the framework of time scale calculus.
Keywords: dynamic delay equation, uniform stability, exponential stability, Lipschitz condition; Lyapunov function.

1. Introduction

The qualitative and quantitative properties of deterministic and stochastic dynamic equations on time scales have received significant attention from many research groups. This interest stems from their critical roles in describing the evolution of ecosystems within random environments, as documented in numerous studies ([1]-[9]). Typically, these investigations are extended to delay dynamic equations because of their importance in describing systems in science and technology, where dynamic of the system state depends not only on the present state but also on its history. Despite their significance, there have been only a few works dealing with delay dynamic equations on time scales. The primary reason is that subtraction on a time scale is generally no longer valid, which raises difficulties in deriving a formal concept of "delay equations on time scales". In [7], [10], [11], the authors have considered some qualitative properties of solutions for deterministic dynamic delay equations. However, the assumptions they imposed on the time scales are very restrictive. To the best of our knowledge, there has not been any study dealing with the qualitative and quantitative properties of stochastic dynamic equations with time-varying delay on time scales. Even in the special case where $\mathbb{T} = \mathbb{R}$, there are only few papers (see [12], [13]) that have proposed stability conditions for stochastic differential equations with time-varying delay via Lyapunov functions, and the conditions derived in them are often difficult to verify. Motivated by these gaps, the purpose of this paper is the following: To propose a viable definition for delay equations on time scales. To consider the existence, uniqueness, and uniform exponential stability of Δ-stochastic dynamic delay equations via the Lyapunov direct method. To improve mathematical techniques, as standard substitution rules in integrals cannot be applied to calculus on time scales, making traditional methods for delay difference/differential equations invalid in this context.

2. Preliminaries and definitions

Let $\mathbb{T}$ be a closed subset of $\mathbb{R}$, enclosed with the topology inherited from the standard topology on $\mathbb{R}$. Let $\sigma(t) = \inf\{s \in \mathbb{T} : s > t\},\ \mu(t) = \sigma(t) - t$ and $\rho(t) = \sup\{s \in \mathbb{T} : s < t\}, \mu(t) = t - \rho(t)$ (supplemented by $\sup \emptyset = \inf \mathbb{T}, \inf \emptyset = \sup \mathbb{T}$). A point $t \in \mathbb{T}$ is said to be *right-dense* if $\sigma(t) = t$, *right-scattered* if $\sigma(t) > t$, *left-dense* if $\rho(t) = t$, *left-scattered* if $\rho(t) < t$ and *isolated* if t is simultaneously right-scattered and left-scattered. The set ${}_k\mathbb{T}$ is defined to be $\mathbb{T}$ if $\mathbb{T}$ does not have a right-scattered minimum; otherwise, it is $\mathbb{T}$ without this right-scattered minimum. A function f defined on $\mathbb{T}$ is *regulated* if there exist a left-sided limit at every left-dense point and a right-sided limit at every right-dense point. A regulated function is called ld*-continuous* if it is continuous at every left-dense point. Similarly, one has the notion of rd-continuous. For $a, b \in \mathbb{T}$, by $[a, b]$, we mean the set $\{t \in \mathbb{T} : a \le t \le b\}$. Denote $\mathbb{T}_a = \{t \in \mathbb{T} : t \ge a\}$ and by $\mathcal{R}$ $(resp.\ \mathcal{R}^+)$ the set of all rd-continuous and regressive (resp. positive regressive) functions. For any function f defined on $\mathbb{T}$, we write f^ρ stands for the function $f \circ \rho$; i.e., $f_t^\rho = f(\rho(t))$ for all $t \in {}_k\mathbb{T}$ and $\lim_{\sigma(s)\uparrow t} f(s)$ by $f(t)$ or f_t if this limit exists. It is easy to see that if t is left-scattered then $f_t = f_t^\rho$. Let $\mathbb{I}$ be the set of all left-scattered points of $\mathbb{T}$. Clearly, $\mathbb{I}$ is at most countable.

Throughout this paper, we suppose that the time scale $\mathbb{T}$ has bounded graininess, that is $\sigma^* = \sup\{\sigma(t) : t \in {}_k\mathbb{T}\} < \infty$.

Let A be an increasing right continuous function defined on $\mathbb{T}$. We denote by μ_Δ^A the Lebesgue Δ-measure associated with A. For any μ_Δ^A-measurable function $f : \mathbb{T} \to \mathbb{R}$, we write $\int_a^t f_s \Delta A_s$ for the integral of f with respect to the measures μ_Δ^A on $(a, t]$. It is seen that the function $t \mapsto \int_a^t f_s \Delta A_s$ is cadlag. It is continuous if A is continuous. In case $A(t) \equiv t$ we write simply $\int_a^t f_s \Delta s$ for $\int_a^t f_s \Delta A_s$. For details, we refer the reader to [6].

Let $p \in \mathcal{R}$ be regulated. We define the so-called exponential function

$$e_p(t, t_0) = \exp\Big\{ \int_{t_0}^t \xi_{\mu(\tau)}(p(\tau))\Delta\tau \Big\}$$

where

$$\xi_h(z) = z \text{ if } h = 0,\ \xi_h(z) = \frac{\mathrm{Ln}(1 + \mathrm{hz})}{h} \ \text{ if } h \ne 0$$

and $\mathrm{Ln}\, x$ to be the principal logarithm of x (see [2, Definition 2.30] in details). Therefore,

$e_p(t, t_0)$ is solution of the initial value problem

$$y^{\Delta}(t) = p(t)y(t), \quad y(t_0) = 1, \quad t > t_0.$$

Define by induction a sequence of polynomial functions $h_k : \mathbb{T} \times \mathbb{T} \to \mathbb{R}$,

$$h_0(t, s) = 1 \text{ and } \quad h_{k+1}(t, s) = \int_s^t h_k(u, s)\Delta u, \quad k \in \mathbb{N}.$$

Since $h_k(t, s)$ is continuous in t, we have $h_{k+1}(t, s) \le \frac{(t-s)^n}{n!}$, $k \in \mathbb{N}$. It is known that (see [1, Theorem 1.113])

$$\sum_{n=0}^{\infty} p^n h_n(t, s) = e_p(t, s). \tag{2.1}$$

Later, we need the following lemma also known as a variant of Gronwall - Bellman inequality

Lemma 2.1 ([3])**.** *Let $u(t)$ be a regulated function and $u_0, \alpha \in \mathbb{R}_+$. Then, the inequality*

$$u(t) \le u_0 + \alpha \int_{t_0}^t u(s)\Delta s \quad \textit{for all} \quad t \in \mathbb{T}_{t_0},$$

implies that $u(t) \le u_0 e_\alpha(t, t_0)$ for all $t \in \mathbb{T}_{t_0}$.

3. Delay dynamic equations on times scales

Let $\mathbb{T}$ be a time scale. There are some works dealing with delay dynamic equations on time scales (see [7, 10, 11]), where the authors attempted to give concepts of delay functions. However, there are only a few time scales matching these concepts. Therefore, based on the varying time bounded delay ideas in differential equations, we define a delay function as an rd-continous map $r(\cdot) :_k \mathbb{T} \to \mathbb{T}$ which satisfies $r(t) \le t$ for all $t \in \mathbb{T}$ and $\tau_* = \sup\{t - r(t) : t \in \mathbb{T}\} < \infty$. For any $s \in \mathbb{T}$, denote $b_s := \min\{r(t) : t \ge s\}$ and consider the set $\Gamma_s = \{r(t) : t \ge s\} \cap [b_s, s]$. We know that $s - b(s) \ge \tau_*$. Let $C(\Gamma_s;\ \mathbb{R}^d)$ be the family of continuous functions from Γ_s to $\mathbb{R}^d$ with the norm $\|\varphi\|_s = \sup_{s\in\Gamma_s} \|\varphi(s)\|$.

Fix $t_0 \in \mathbb{T}$ and consider the Δ-delay equations on a time scale

$$\begin{cases} X^{\Delta}(t) = f\big(t, X(t), X(r(t))\big), t \in \mathbb{T}_{t_0}, \\ X(s) = \xi(s), \quad \forall\ s \in \Gamma_{t_0}, \end{cases} \tag{3.1}$$

where $f : \mathbb{T} \times \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^d$ is a continuous function and $\xi = \{\xi(s) :\ s \in \Gamma_{t_0}\}$ is in $C(\Gamma_{t_0})$. In the following we

denote by $\widetilde{\mathbb{T}}_s$ the set $\Gamma_s \cup \mathbb{T}_s$ for any $s \in \mathbb{T}$.

Definition 3.1. *A function $(X(t))_{t\in\widetilde{\mathbb{T}}_{t_0}}$, valued in $\mathbb{R}^d$, is called a solution of equation (3.1) if it is Δ- differentiable and* (3.1) *holds.*

The equation (3.1) is said to have the uniqueness of solutions if $X(t)$ and $\overline{X}(t)$ with $X(t) = \overline{X}(t)$ for $t \in \Gamma_{t_0}$ are two processes satisfying (3.2) then

$$X(t) = \overline{X}(t), \text{ for all } t \in \mathbb{T}_{t_0}.$$

Since f is continuous, $X(t)$ is a solution of (3.1) if and only if

$$X(t) = \xi(t_0) + \int_{t_0}^{t} f\big(s, X(s), X(r(s))\big)\Delta s, \quad t \geq t_0. \tag{3.2}$$

We now give conditions guaranteeing the existence and uniqueness of solutions to equation (3.1). First, we consider the case where the coefficients satisfy Lipschitz and linear growth conditions.

Theorem 3.1 (Existence and uniqueness of solution)**.** *Assume that for any $T \in \mathbb{T}_{t_0}$, there exist positive constants $\kappa = \kappa(T)$ and $\bar{\kappa} = \bar{\kappa}(T)$ such that*

(i) *(Lipschitz condition) for all $x_i, y_i \in \mathbb{R}^d, i = 1, 2$ and $t \in [t_0, T]$,*

$$\|f(t, x_1, y_1) - f(t, x_2, y_2)\| \leq \kappa\big(\|x_1 - x_2\| + \|y_1 - y_2\|\big). \tag{3.3}$$

(ii) *(Linear growth condition) for all $(t, x, y) \in [t_0, T] \times \mathbb{R}^d \times \mathbb{R}^d$,*

$$\|f(t, x, y)\| \leq \bar{\kappa}\big(1 + \|x\| + \|y\|\big). \tag{3.4}$$

Then, there exists a unique solution $X(t)$ to equation (3.1).

Proof. For the existence of a solution, let $T \in \mathbb{T}_{t_0}$ and we will show that a solution of equation (3.1) exists on $[t_0, T]$. We set

$$X_0(t) = \xi(t), t \in \Gamma_{t_0} \quad \text{and} \quad X_0(t) = \xi(t_0), \ t \in \mathbb{T}_{t_0}.$$

Based on (3.2), we define the Picard iteration: $X_n(t) = \xi(t), \ t \in \Gamma_{t_0}$ and

$$X_n(t) = \xi(t_0) + \int_{t_0}^{t} f\big(s, X_{n-1}(s), X_{n-1}(r(s))\big)\Delta s. \tag{3.5}$$

Since $s \in [t_0, s]$, $r(s) \subset \Gamma_{t_0} \cup [t_0, r(s)]$ and $\|X_n(\tau) - X_{n-1}(\tau)\| = 0$ on Γ_{t_0},

$$\begin{aligned}
\|X_{n+1}(t) - X_n(t)\| &\leq \int_{t_0}^{t} \big\| f\big(s, X_n(s), X_n(r(s))\big) - f\big(s, X_{n-1}(s), X_{n-1}(r(s))\big)\big\| \Delta s \\
&\leq \kappa \int_{t_0}^{t} \Big(\|X_n(s) - X_{n-1}(s)\| + \|X_n(r(s)) - X_{n-1}(r(s))\|\Big)\Delta s \\
&\leq \kappa \int_{t_0}^{t} \Big(\sup_{t_0 \leq \tau \leq s} \|X_n(\tau) - X_{n-1}(\tau)\| + \sup_{\tau \in \Gamma_{t_0} \cup [t_0, r(s)]} \|X_n(\tau) - X_{n-1}(\tau)\|\Big)\Delta s \\
&\leq \kappa \int_{t_0}^{t} \Big(\sup_{t_0 \leq \tau \leq s} \|X_n(\tau) - X_{n-1}(\tau)\| + \sup_{\tau \in [t_0, r(s)]} \|X_n(\tau) - X_{n-1}(\tau)\|\Big)\Delta s \\
&\leq 2\kappa \int_{t_0}^{t} \sup_{t_0 \leq \tau \leq s} \|X_n(\tau) - X_{n-1}(\tau)\| \Delta s.
\end{aligned}$$

Let $\sup_{t_0\le s\le T}\|X_1(s)-X_0(s)\|=C$. By induction, we can prove that

$$\sup_{t_0\le s\le t}\|X_{n+1}(s)-X_n(s)\|\le C(2\kappa)^n h_n(t,t_0). \tag{3.6}$$

Since $\sum_{n=0}^{\infty}(2\kappa)^n h_n(T,t_0)=e_{2\kappa}(T,t_0)$, it follows that $X_n(\cdot)$ is a Cauchy sequence. Hence, there exists a function $X(t)$ such that

$$\lim_{n\to\infty}\sup_{t_0\le t\le T}\|X_n(t)-X(t)\|=0.$$

Let $n\to\infty$ in (3.5), we see that $X(t)$ satisfies Equation (3.2). Further, let $X(t)$ and $\overline{X}(t)$ be two solutions of equation (3.1). Then, by a similar argument as above we have

$$\sup_{t_0\le\tau\le t}\|X(\tau)-\bar{X}(\tau)\|\le 2\kappa\int_{t_0}^{t}\sup_{t_0\le\tau\le s}\|X(\tau)-\bar{X}(\tau)\|\Delta s.$$

Using Gronwall-Bellman inequality yields

$$\sup_{t_0\le s\le T}\|X(\tau)-\overline{X}(\tau)\|=0,$$

i.e., $X(t)=\overline{X}(t)$ for all $t_0\le t\le T$. The uniqueness of the solution has been proved. The proof is complete. □

In the proof of Theorem 3.1, we have shown that the Picard iterative sequence $X_n(t)$ converges to the unique solution $X(t)$ of equation (3.1). The following theorem gives an estimate on the rate of convergence.

Theorem 3.2. *Assume the assumptions of Theorem 3.1 hold. Let $X(t)$ be the unique solution of equation (3.1) and $X_n(t)$ be the Picard iteration defined by (3.8). Then,*

$$\sup_{t_0\le t\le T}\|X_n(t)-X(t)\|\le C(2\kappa)^n h_n(T,t_0)e_{2\kappa}(T,t_0), \tag{3.7}$$

for all $n\ge 1$.

Proof. By using similar arguments in the proof of 3.1,

$$\|X_n(t)-X(t)\|\le\kappa\int_{t_0}^{t}\Big(\|X_{n-1}(s)-X(s)\|+\|X_{n-1}(r(s))-X(r(s))\|\Big)\Delta s.$$

Therefore,

$$\begin{aligned}\sup_{t_0\le s\le t}\|X_n(s)-X(s)\|\le{}& 2\kappa\int_{t_0}^{t}\sup_{t_0\le\tau\le s}\|X_{n-1}(\tau)-X(\tau)\|\Delta s\\ &\le 2\kappa\int_{t_0}^{T}\sup_{t_0\le\tau\le s}\big\{\|X_n(\tau)-X_{n-1}(\tau)\|+\|X_n(\tau)-X(\tau)\|\big\}\Delta s\\ &\overset{(3.6)}{\le} C(2\kappa)^n h_n(t,t_0)+2\kappa\int_{t_0}^{t}\sup_{t_0\le\tau\le s}\|X_n(\tau)-X(\tau)\|\}]\Delta s.\end{aligned}$$

By using the Gronwall-Bellman inequality we get

$$\sup_{t_0 \le s \le t} \|X_n(s) - X(s)\| \le C(2\kappa)^n h_n(t, t_0) e_{2\kappa}(t, t_0).$$

That we get $\sup_{t_0 \le t \le T} \|X_n(t) - X(t)\| \le C(2\kappa)^n h_n(T, t_0) e_{2\kappa} T, t_0)$. The proof is complete. □

We now consider the case where the coefficients of equation (3.1) are locally Lipschitz. Let $C^{1,1}\big([a,b]\times\mathbb{R}^d;\mathbb{R}\big)$ be the set of all functions $V(t,x)$ defined on $[a,b]\times\mathbb{R}^d$, having continuous Δ-derivative in t and x. Let

$$\mathbb{I} = \{t \in \mathbb{T}_{t_0} : t \text{ is } \mu(t) > 0\}.$$

For any $V \in C^{1,1}(\mathbb{T}_{t_0} \times \mathbb{R}^d;\mathbb{R})$ define

$$\begin{aligned} \mathcal{L}V(t,x,y) = V^{\Delta_t}(t,x) + \sum_{i=1}^{d} \frac{\partial V(t,x)}{\partial x_i}\big(1 - 1_{\mathbb{I}}(t)\big) f_i(t,x,y) \\ + \Big(V\big(t, x + f(t,x,y)\mu(t)\big) - V(t,x)\Big)\Phi(t), \end{aligned} \tag{3.8}$$

where V^{Δ_t} is partial Δ-derivative of $V(t,x)$ in t and $\Phi(t) = \begin{cases} 0 & \text{if } \mu(t) = 0, \\ \frac{1}{\mu(t)} & \text{if } \mu(t) > 0. \end{cases}$

Theorem 3.3. *Suppose that for any $k > 0$ and $T \in \mathbb{T}_{t_0}$, there exists a constant $L_{T,k} > 0$ such that*

$$\|f(t,x_1,y_1) - f(t,x_2,y_2)\| \le L_{T,k}\big(\|x_1 - x_2\| + \|y_1 - y_2\|\big), \tag{3.9}$$

for all $x_i, y_i \in \mathbb{R}^d,\ i = 1,2$, with $\|x_i\| \vee \|y_i\| \le k$ and $t \in [t_0, T]$. Further, there are two positive constants λ_1, λ_2 and a function $V \in C^{1,1}\big([b_{t_0}, T] \times \mathbb{R}^d;\mathbb{R}_+\big)$ satisfying

$$\mathcal{L}V(t,x,y) \le \lambda_1 V(t,x) + \lambda_2 V\big(r(t), y\big), \tag{3.10}$$

and $\lim_{\|x\|\to\infty} \inf_{t\in[t_0,T]} V(t,x) = \infty$. Then, the equation (3.1) *has a unique solution $X(t)$ defined on $\widetilde{\mathbb{T}}_{t_0}$.*

Proof. For each $k \ge k_0 = [\|\xi\|_{t_0}] + 1$, define the truncation function

$$f_k(t,x,y) = \begin{cases} f(t,x,y) & \text{if } \|x\| \vee \|y\| \le k \\ f\Big(t, \frac{kx}{\|x\|}, \frac{ky}{\|y\|}\Big) & \text{if } \|x\| \vee \|y\| > k. \end{cases}$$

It is easy to see that f_k is bounded and satisfies the global Lipschitz condition. Therefore, by Theorem 3.1 there exists a unique solution $X_k(\cdot)$ to the equation

$$\begin{cases} X^{\Delta}(t) = f_k\big(t, X(t), X(r(t))\big) \\ X(s) = \xi(s) \in \mathbb{R}^d,\ \forall\, s \in \Gamma_{t_0}. \end{cases} \tag{3.11}$$

Define a sequence of time

$$\theta_k = \inf\{t \in \mathbb{T}_{t_0} : \|X_k(t)\| \geq k\}, \quad \theta_{k_0} = t_0.$$

It is easy to see that θ_k is increasing and we have

$$X_k(t) = X_{k+1}(t) \quad \text{if} \;\; t_0 \leq t \leq \theta_k. \tag{3.12}$$

Denote $\theta_\infty = \lim\limits_{k\to\infty} \theta_k$ and define the function $X(t),\; t_0 \leq t \leq \theta_\infty$ by

$$X(t) = X_k(t), \;\; t_0 \leq t < \theta_k, \;\; k \geq k_0.$$

Using (3.12) gets $X(\theta_k \wedge t) = X_k(\theta_k \wedge t)$. By (3.10) it yields

$$\begin{aligned} V\big(\theta_k \wedge t, X(\theta_k \wedge t)\big) &= V(t_0, \xi(t_0)) + \int_{t_0}^{t} \mathcal{L}V\big(s, X(s), X(r(s))\big)(s) 1_{[t_0,\theta_k]}(s)\Delta s \\ &\leq V\big(t_0, \xi(t_0)\big) + \int_{t_0}^{t} \Big(\lambda_1 V(s, X(s)) + \lambda_2 V(r(s), X(r(s)))\Big) 1_{[t_0,\theta_k]}(s)\Delta s. \end{aligned}$$

Let $s_0 = \inf\{t : r(t) \geq t_0\}$. We have

$$\begin{aligned} &\sup_{t_0\leq s\leq t} V\big(\theta_k \wedge s, X(\theta_k \wedge s)\big) \leq V\big(t_0, \xi(t_0)\big) \\ &\quad + \lambda_1 \int_{t_0}^{t} V(s, X(s)) 1_{[t_0,\theta_k]}(s)\Delta s + \lambda_2 \int_{t_0}^{t} V(r(s), X(r(s))) 1_{[t_0,\theta_k]}(s)\Delta s \\ &\quad = V\big(t_0, \xi(t_0)\big) + \lambda_1 \int_{t_0}^{t} V\big(s, X(s)\big) 1_{[t_0,\theta_k]}(s)\Delta s \\ &\quad + \lambda_2 \int_{t_0}^{s_0} V\big(r(s), X(r(s)\big) 1_{[t_0,\theta_k]}(s)\Delta s + \lambda_2 \int_{s_0}^{t} V\big(r(s), X(r(s))\big) 1_{[t_0,\theta_k]}(s)\Delta s \\ &\quad \leq C_2 + \lambda_1 \int_{t_0}^{t} \sup_{t_0\leq\tau\leq s} V\big(\tau, X(\tau)\big) 1_{[t_0,\theta_k]}(s)\Delta s + \lambda_2 \int_{t_0}^{t} \sup_{t_0\leq\tau\leq r(s)} V\big(\tau, X(\tau)\big) 1_{[t_0,\theta_k]}(s)\Delta s \\ &\quad \leq C_2 + \lambda \int_{t_0}^{t} \sup_{t_0\leq\tau\leq s} V\big(\theta_k \wedge \tau, X(\theta_k \wedge \tau)\big)\Delta s, \end{aligned}$$

where $\lambda = \lambda_1 + \lambda_2$ and

$$C_2 = V\big(t_0, \xi(t_0)\big) + \lambda_2 \int_{t_0}^{s_0} V\big(r, \xi(r)\big)(s)\Delta s.$$

Using the Gronwall-Bellman inequality yields

$$\sup_{t_0\leq s\leq t} V(\theta_k \wedge s, X(\theta_k \wedge s)) \leq C_2\, e_\lambda(t, t_0).$$

Hence,

$$V\big(\theta_k \wedge t, X(\theta_k \wedge t)\big) \le C_2\, e_\lambda(t, t_0).$$

On the other hand, if $\theta_\infty < T$, follow the definition of θ_k, we have

$$\limsup_{t \to \theta_\infty} \|X(t)\| = \infty.$$

Therefore, the property $\lim\limits_{\|x\|\to\infty} \inf\limits_{t\in[t_0,T]} V(t,x) = \infty$, which leads to a contradiction. Thus, $\theta_\infty > T$, i.e., the solution $X(t)$ is defined on $[t_0, T]$. The uniqueness follows immediately from the uniqueness of solutions of equation (3.11). □

4. On the stability of delay dynamic equation

In this section we give sufficient conditions for the exponential stability of equation (3.1). We suppose that for any $s > t_0$ and $\xi \in C(\Gamma_s;\ \mathbb{R}^d)$, there exists a unique solution $X(t, s, \xi), t \in \mathbb{T}_s$ of the equation (3.1) satisfying $X(t, s, \xi) = \xi(t)$ for any $t \in \Gamma_s$. Furthermore,

$$f(t, 0, 0) \equiv 0;\ g(t, 0, 0) \equiv 0,\ \ \forall\, t \in \mathbb{T}_{t_0}. \tag{4.1}$$

From the condition (4.1), equation (3.1) has a trivial solution $X(t, s, 0) \equiv 0$.

Definition 4.1. *The trivial solution $X(t, s, 0) \equiv 0$ of equation* (3.1) *is said to be exponentially stable if there is a positive constant α such that for any $s > t_0$, there exists $\beta_s > 0$ for which the following relation*

$$\|X(t, s, \xi)\| \le \beta_s \|\xi\|_s e_{\ominus\alpha}(t, s)\ \ \text{on}\ \ t \ge s, \tag{4.2}$$

holds for any $\xi \in C(\Gamma_s;\ \mathbb{R}^d)$.

If one can choose β_s independent of s, the trivial solution of the equation (3.1) is called uniformly exponentially stable.

Theorem 4.1. *Let $\alpha_1, \alpha_2, p, c_1, c_2$ be positive numbers with $\alpha_1 > \alpha_2$. Suppose that there exists a positive definite function $V \in C^{1,1}(\mathbb{T}_{t_0} \times \mathbb{R}^d; \mathbb{R}_+)$ such that*

$$c_1\|x\|^p \le V(t, x) \le c_2\|x\|^p \quad \forall (t, x) \in \mathbb{T}_{t_0} \times \mathbb{R}^d, \tag{4.3}$$

and for all $(t, x, y) \in \mathbb{T}_{t_0} \times \mathbb{R}^d \times \mathbb{R}^d$

$$\mathcal{L}V(t, x, y) \le -\frac{\alpha_1}{1 + \alpha_1\mu(t)} V(t, x) + \frac{\alpha_2 e_{\ominus\alpha_1}(t, r(t))}{1 + \alpha_2\mu(t)} V(r(t), y), \tag{4.4}$$

the equation (3.1) *is uniformly exponentially stable.*

Proof. Let $s > t_0$ and $X(t) = X(t, s, \xi)$ be the solution of the equation (3.1) with the initial condition $X(t) = \xi(t)$ for all $t \in \Gamma_s$. By (4.4) we get

$$\begin{aligned}
&e_{\alpha_1}(t,s)V(t,X(t))\\
&= V(s,\xi(s)) + \int_s^t e_{\alpha_1}(\tau,s)\Big[\alpha_1 V\big(\tau,X(\tau)\big) + (1{+}\alpha_1\mu(\tau))\mathcal{L}V\big(\cdot,X,X(r)\big)(\tau)\Big]\Delta\tau\\
&\le V(s,\xi(s)) + \int_s^t e_{\alpha_1}(\tau,s)\Big[\alpha_1 V(\tau,X(\tau))\\
&\quad + (1+\alpha_1\nu(\tau))\Big(\frac{-\alpha_1}{1+\alpha_1\mu(\tau)}V(\tau,X(\tau)) + \frac{\alpha_2 e_{\ominus\alpha_1}(\tau,r(\tau))}{1+\alpha_2\mu(\tau)}V\big(r,X(r)\big)(\tau)\Big)\Big]\Delta\tau\\
&\le V(s,\xi(s)) + \int_s^t \frac{\alpha_2(1+\alpha_1\mu(\tau))e_{\ominus\alpha_1}(\tau,r(\tau))e_{\alpha_1}(\tau,s)}{1+\alpha_2\mu(\tau)}V\big(r,X(r)\big)(\tau)\Delta\tau.
\end{aligned}$$

Since the function $\frac{\alpha_2(1+\alpha_1 x)}{1+\alpha_2 x}$ is increasing in x and $\lim\limits_{x\to\infty}\frac{\alpha_2(1+\alpha_1 x)}{1+\alpha_2 x} = \alpha_1$,

$$\frac{\alpha_2(1+\alpha_1\mu(\tau))}{1+\alpha_2\mu(\tau)} \le \frac{\alpha_2(1+\alpha_1\mu_*)}{1+\alpha_2\mu_*} =: \alpha_3 < \alpha_1.$$

Further, by [1, Theorem 2.36, pp. 62],

$$e_{\ominus\alpha_1}(\tau,r(\tau))e_{\alpha_1}(\tau,s) = e_{\alpha_1}(r(\tau),s).$$

Therefore, with $t \ge s$,

$$\begin{aligned}
&\sup_{s\le\tau\le t} e_{\alpha_1}(\tau,s)V(\tau,X(\tau)) \le V(s,\xi(s)) + \alpha_3\int_s^t e_{\alpha_1}(r(\tau),s)V(r(\tau),X(r(\tau)))1_{[s,\theta_n]}(\tau)\Delta\tau\\
&= V(s,\xi(s)) + \alpha_3\int_s^t [e_{\alpha_1}(r(\tau),s)V(r(\tau),X(r(\tau)))]\Delta\tau\\
&\le V(s,\xi(s)) + \alpha_3(s-b_s)\sup_{b_s\le u\le s} e_{\alpha_1}(u,s)V(u,\xi(u)) + \alpha_3\int_s^t \sup_{s\le u\le\tau}[e_{\alpha_1}(u,s)V(u,X(u))]\Delta s.
\end{aligned}$$

We see that

$$V(s,\xi(s)) + \alpha_3(s-b_s)\sup_{b_s\le u\le s} e_{\alpha_1}(u,s)V(u,\xi(u)) \le c_2(1+\alpha_3\tau_* e^{\alpha_1\tau_*})\|\xi\|_s^p.$$

Let $c_3 = c_2(1+\alpha_3\tau_* e^{\alpha_1\tau_*})$. Using Gronwall-Bellman inequality yields

$$\sup_{s\le u\le t} e_{\alpha_1}(u,s)V(u,X(u)) \le c_3 e_{\alpha_3}(t,s)\|\xi\|_s^p.$$

By virtue of Fatou's Lemma, we can take $n\to\infty$ to obtain

$$\sup_{s\le u\le t} e_{\alpha_1}(u,s)V(u,X(u)) \le c_3 e_{\alpha_3}(t,s)\|\xi\|_s^p.$$

Hence,

$$c_1\|X(t)\|^p \le V(t,X(t)) \le c_3\|\xi\|_s^p \frac{e_{\alpha_3}(t,s)}{e_{\alpha_1}(t,s)} \le c_3 e_{\ominus\alpha}(t,s)\|\xi\|_s^p,$$

where $\alpha = \alpha_3 \ominus \alpha_1 = \frac{\alpha_1-\alpha_3}{1+\alpha_3\nu_*}$. Thus,

$$\|X(t)\|^p \le \frac{c_3}{c_1} e_{\ominus\alpha}(t,s)\|\xi\|_s^p.$$

Since $e_{\ominus\alpha}(t,t_0) = \exp\Big\{\int_{t_0}^t \lim_{h\searrow\mu(s)} \frac{\mathrm{Ln}(1+[\ominus\alpha]h)}{h}\Delta s\Big\}$, by a direct computation we can choose a sufficiently small positive number $\beta = \frac{(1+\alpha\mu^*)^{\frac{1}{p}}-1}{\mu^*} \in \mathcal{R}^+$ such that which implies that

$$(e_{\ominus\alpha}(t,s))^{\frac{1}{p}} \le e_{\ominus\beta}(t,s).$$

Hence, there exits $\beta_s = \big(\frac{c_3}{c_1}\big)^{\frac{1}{p}}$ such that

$$\|X(t)\| \le \beta_s e_{\ominus\beta}(t,s)\|\xi\|_s.$$

This means that the equation (3.1) is uniformly exponentially stable. The proof is complete. □

Example 4.1. *Let $\mathbb{T}$ be a time scale containing 0 and $r(t)$ be a delay function. Assume that A and B are $d\times d$ matrices, we consider the dynamic delay equation on time scale $\mathbb{T}$*

$$\begin{cases} X^\Delta(t) = AX(t) + BX(r(t)) \\ X(s) = \xi(s) \ \ \forall\, s \in \Gamma_0, t \in \mathbb{T}_0. \end{cases} \tag{4.5}$$

By using $V(t,x) = \|x\|^2 = x^\top x$ we have $V^{\Delta_t}(t,x) = 0$. Thus

$$\begin{aligned}\mathcal{L}V(t,x,y) &= \sum_{i=1}^d \frac{\partial V(t,x)}{\partial x_i} f_i(t,x,y) = 2\langle x^\top, Ax + By\rangle \\ &= x^\top(A+A^\top)x + x^\top By + y^\top B^\top x = 2x^\top Ax + 2x^\top By\end{aligned}$$

if $t \notin \mathbb{I}$, and when $t \in \mathbb{I}$

$$\begin{aligned}\mathcal{L}V(t,x,y) &= \frac{V(t,x+f(t,x,y)\mu(t)) - V(t,x)}{\mu(t)} = \frac{\|x+\mu(Ax+By)\|^2 - \|x\|^2}{\mu(t)} \\ &= \frac{1}{\mu(t)}\Big[\langle x^\top + \mu x^\top A^\top + \mu y^\top B^\top, x + Ax + By\rangle - \langle x^\top, x\rangle\Big] \\ &= x^\top\big(A^\top + A + \mu(t)AA^\top\big)x + x^\top By + y^\top Bx + \mu(t)x^\top A^\top By \\ &\quad + \mu(t)y^\top B^\top Ax + \mu(t)y^\top B^\top By \\ &= x^\top\big(A^\top + A + \mu AA^\top\big)x + x^\top(B+B^\top)y + \mu(t)x^\top(A^\top B + A^\top B)y \\ &\quad + \mu(t)y^\top B^\top By.\end{aligned}$$

Suppose that the abscissa of the matrix $A^\top + A + AA^\top\mu(t)$ is uniformly bounded by a negative constant $-\alpha_1$. Then,

$$\begin{aligned}\mathcal{L}V(t,x,y) &\le -\alpha_1\|x\|^2 + 2\big(1+\mu(t)\|A\|\big)\|B\|\|x\|\|y\| + \mu(t)\|B\|^2\|y\|^2\\ &\le -\frac{\alpha_1}{1+\alpha_1\mu(t)}\|x\|^2 + \frac{\|B\|e_{\ominus\alpha_1}(t,r(t))}{1+\mu(t)\|B\|}\|y\|^2 - \Big[\frac{\mu(t)\alpha_1^2}{1+\alpha_1\mu(t)}\|x\|^2\\ &\quad + \Big(\frac{\|B\|e_{\ominus\alpha_1}(t,r(t))}{1+\|B\|\mu(t)} - \mu(t)\|B\|^2\Big)\|y\|^2 - 2(1+\mu(t)\|A\|)\|B\|\|x\|\|y\|\Big].\end{aligned}$$

Since

$$e^{-\alpha_1\mu^*} \le e^{-\alpha_1(t-r(t)} \le e_{\ominus\alpha_1}(t,r(t)),\ \text{for all } t\in\mathbb{T}_s,$$

it is easy to see that if $\|B\|$ is sufficiently small, then

$$\begin{aligned}\frac{\mu(t)\alpha_1^2}{1+\alpha_1\mu(t)}\|x\|^2 + \Big(\frac{\|B\|e_{\ominus\alpha_1}(t,r(t))}{1+\|B\|\mu(t)} - \mu(t)\|B\|^2\Big)\|y\|^2 \\ - 2(1+\mu(t)\|A\|)\|B\|\|x\|\|y\| \ge 0,\end{aligned} \tag{4.6}$$

which implies that

$$\mathcal{L}V(t,x,y) \le -\frac{\alpha_1}{1+\alpha_1\mu(t)}\|x\|^2 + \frac{\|B\|e_{\ominus\alpha_1}(t,r(t))}{1+\|B\|\mu(t)}\|y\|^2.$$

Therefore, the assumptions of Theorem 4.1 are satisfied, which ensure the trivial solution of equation (4.5) is exponentially stable.

Let $\mathbb{T}$ be a time scale defined by $\mathbb{T} = \bigcup_{k=1}^{\infty}\left[\frac{5k}{4}, \frac{5k+4}{4}\right]$. Consider the equation (4.5) with $r(t) = t_-$ and

$$A = \begin{pmatrix} -1 & 0 & 0\\ 0 & -1 & -0.3\\ 0 & 0.5 & -1\end{pmatrix}, \qquad B = \begin{pmatrix} 0.1 & 0.05 & 0\\ 0.03 & 0.02 & -0.05\\ 0 & 0.05 & 0.03\end{pmatrix}.$$

In this case, $\mu^* = \frac{1}{4}$, $\mu(t) = 0$ when $t \in \left(\frac{5k}{4}, \frac{5k+4}{4}\right]$ and $\mu(t) = \frac{1}{4}$ when $t = \frac{5k}{4}$. Furthermore, the abscissa of the matrix $A^\top + A + AA^\top\mu(t)$ is uniformly bounded by a negative constant $-\alpha_1 = -1.56$; $\|A\| = 1,5$, $\|B\| = 0.06$. Therefore, the inequality (4.6) is satisfied, which confirms that the system (4.5) is uniformly exponentially stable.

Note for contributor:

- Short bio: Nguyen Thu Ha is a lecturer at the Faculty of Natural Sciences, Electric Power University.

- Author's contributions: Nguyen Thu Ha: conceptualization, methodology, data analysis, writing & editing.

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