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COMPOSITION OPERATORS ON PLURIHARMONIC HARDY SPACES ON THE UNIT BIDISC

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Abstract. In this paper, we study composition operators induced by holomorphic maps $\Phi = (\phi, \psi) : \mathbb{D} \to \mathbb{D}^2$ acting from the pluriharmonic Hardy space $H^2_h(\mathbb{D}^2)$ into $H^2_h(\mathbb{D})$. Using a structural decomposition of pluriharmonic functions on the bidisc into holomorphic and anti-holomorphic parts, we reduce the underlying problem to estimates on classical holomorphic Hardy spaces. These results extend the corresponding holomorphic theory to the pluriharmonic Hardy setting on the bidisc.
Keywords: composition operators, Hardy spaces, analytic functions, pluriharmonic functions.

1. Introduction

Composition operators constitute a central class of operators on spaces of holomorphic and harmonic functions. Let $\mathbb{D}$ denote the unit disk in $\mathbb{C}$ and $\mathbb{D}^2$ the bidisc in $\mathbb{C}^2$. Given a holomorphic map $\Phi = (\varphi, \psi) : \mathbb{D} \to \mathbb{D}^2$, one naturally associates the composition operator acting on functions on the bidisc by

$$(C_\Phi f)(z) = f(\varphi(z), \psi(z)), \qquad z \in \mathbb{D}.$$

In the holomorphic setting, such operators between Hardy and Bergman-type spaces have been studied extensively, including the problem of determining when symbols acting between domains of different dimensions induce bounded or compact composition operators. For more details, the readers may consult [1]-[3]

To the best of our knowledge, composition operators between harmonic or pluriharmonic Hardy spaces have received relatively little attention (see [4] and the references therein). Nevertheless, several recent works indicated that the study of composition operators in harmonic and pluriharmonic settings is becoming increasingly active. For example, Chen and Hamada investigated composition operators on Bloch and Hardy type spaces and obtained characterizations involving harmonic and pluriharmonic function spaces [5]. They also developed Hardy-space and composition-operator theory for holomorphic and pluriharmonic functions on bounded symmetric domains [6]. In a related direction, Kosiński studied composition operators on the polydisc and established boundedness results for Hardy spaces and weighted Bergman spaces over $\mathbb{D}^n$ [7]. These developments show that operator-theoretic questions in several complex variables and in non-holomorphic function spaces continue to attract attention.

The present paper contributes to this line of research by focusing on the specific operator $C_\Phi : H^2_h(\mathbb{D}^2) \to H^2_h(\mathbb{D})$, thereby providing a pluriharmonic counterpart on the bidisc to the holomorphic framework studied in [1].

Our motivation comes from the holomorphic theory developed by Izuchi–Nguyen–Ohno [1], where composition operators induced by holomorphic maps from $\mathbb{D}$ to $\mathbb{D}^2$ were analyzed on holomorphic Hardy spaces. In particular, [1] provides a simple sufficient condition for boundedness analogous to a uniform "separation from the distinguished boundary" via the product $\varphi\psi$, and a compactness result in which compactness of the one-variable composition operators C_φ and C_ψ plays a decisive role. One of our main objectives is to show that, after incorporating the intrinsic structure of pluriharmonic Hardy functions on the bidisc, these mechanisms persist and yield corresponding boundedness and compactness theorems for C_Φ acting between $H^2_h(\mathbb{D}^2)$ and $H^2_h(\mathbb{D})$.

The key additional ingredient in the pluriharmonic setting is a structural decomposition of functions in $H^2_h(\mathbb{D}^2)$, which allows us to reduce questions about C_Φ to estimates involving holomorphic Hardy data (and their conjugates). Combining this decomposition with the strategy of [1], we establish two theorems that parallel Theorem 2.1 and Theorem 3.1 of [1]. The paper is organized as follows. Section 2 collects preliminaries on pluriharmonic Hardy spaces and proves a structural proposition describing the form of functions in $H^2_h(\mathbb{D}^2)$ (a bidisc analogue of the holomorphic expansion used in [1]). In Section 3 we study boundedness and compactness of $C_\Phi : H^2_h(\mathbb{D}^2) \to H^2_h(\mathbb{D})$. Our arguments develop in close analogy with the approach of [1], and the proofs combine the two-variable symbol conditions (as in the holomorphic theory) with the structural proposition from Section 2 in order to handle the pluriharmonic nature of the functions involved.

2. Preliminaries

Definition 2.1. *Let u be a $\mathcal{C}^2$ smooth function defined on a domain $D \in \mathbb{C}^n$. We say that u is pluriharmonic if $dd^c u = 0$, or equivalently,*

$$\frac{\partial^2 u}{\partial z_j \partial \bar{z}_k} = 0, \ \forall 1 \leq j, k \leq n.$$

In analogy to holomorphic Hardy spaces on polydisc, we have the following notion of pluriharmonic Hardy spaces as defined in [8].

Definition 2.2 (Pluriharmonic Hardy space on the polydisc)**.** *Let $\mathbb{D}^n \subset \mathbb{C}^n$ be the unit polydisc and let $\mathbb{T}^n$ be its distinguished boundary, equipped with normalized Haar measure m_n. For $0 < p < \infty$, the* pluriharmonic Hardy space $H^p_h(\mathbb{D}^n)$ *consists of all pluriharmonic functions u on $\mathbb{D}^n$ such that*

$$\|u\|^p_{H^p_h(\mathbb{D}^n)} := \sup_{0<r<1} \int_{\mathbb{T}^n} \big|u(r\zeta)\big|^p \, dm_n(\zeta) < \infty,$$

where $r\zeta = (r\zeta_1, \ldots, r\zeta_n)$ for $\zeta \in \mathbb{T}^n$.

For $p = 2$, $H^2_h(\mathbb{D}^n)$ becomes a Hilbert space with inner product

$$\langle u, v\rangle_{H^2_h(\mathbb{D}^n)} := \lim_{r\uparrow 1} \int_{\mathbb{T}^n} u(r\zeta) \, v(r\zeta) \, dm_n(\zeta),$$

where the limit exists for $u, v \in H^2_h(\mathbb{D}^n)$.

We also write $\mathcal{O}(\Omega)$ for the space of holomorphic functions on a domain $\Omega \subset \mathbb{C}^n$.

As mentioned in the introduction, for ease of exposition we restrict our attention to pluriharmonic Hardy spaces on the bidisc. Our first result provides a useful bridge between pluriharmonic functions and holomorphic functions on $\mathbb{D}^2$.

Proposition 2.1. *Let u be a pluriharmonic function on the bidisc $\mathbb{D}^2$. Then there exist holomorphic functions f and g on $\mathbb{D}^2$ such that $u = f + \bar{g}$ on $\mathbb{D}^2$.*

For the proof, we need the following classical fact about exact differential forms on contractible domains in $\mathbb{R}^m$.

Lemma 2.1 (Poincaré lemma for contractible domains)**.** *Let $\Omega \subset \mathbb{R}^m$ be a contractible open set. If β is a C^1 closed 1-form on Ω (i.e. $d\beta = 0$), then β is exact: there exists a C^2 function F on Ω such that $dF = \beta$.*

Proof of Proposition 2.1. Write $z = (z_1, z_2)$ for the standard coordinates on $\mathbb{C}^2$. Consider the $(1,0)$-form

$$\alpha := \partial u = \frac{\partial u}{\partial z_1} \, dz_1 + \frac{\partial u}{\partial z_2} \, dz_2.$$

We show that α is closed. Indeed, since $u \in C^2$, mixed derivatives commute by Schwarz's theorem; hence

$$\frac{\partial}{\partial z_2}\left(\frac{\partial u}{\partial z_1}\right) = \frac{\partial}{\partial z_1}\left(\frac{\partial u}{\partial z_2}\right).$$

This is precisely the condition $d\alpha = 0$ (equivalently, $\partial\alpha = 0$ for a $(1,0)$-form). Next, we observe that the bidisc $\mathbb{D}^2$ is contractible (hence, in particular, simply connected), so by Lemma 2.1 there exists a C^2 function F on $\mathbb{D}^2$ such that

$$dF = \alpha.$$

We now show that F is holomorphic. Decompose $dF = \partial F + \bar{\partial} F$ by type. Since α is of pure type $(1,0)$, it has no $(0,1)$-component. Therefore the $(0,1)$-part of $dF = \alpha$ yields

$$\bar{\partial} F = 0.$$

Hence F is holomorphic on $\mathbb{D}^2$. Set $f := F \in \mathcal{O}(\mathbb{D}^2)$. Finally, we let $h := u - f$. Then

$$\partial h = \partial u - \partial f = \alpha - \partial f.$$

But from $df = \alpha$ we have in particular $\partial f = \alpha$, so $\partial h = 0$. Thus h is anti-holomorphic, i.e. there exists $g \in \mathcal{O}(\mathbb{D}^2)$ with $h = \overline{g}$. Consequently,

$$u = f + \overline{g} \quad \text{on } \mathbb{D}^2,$$

as desired. $\square$

Proposition 2.2. *Let u be pluriharmonic on $\mathbb{D}^2$, and suppose that*

$$u = f + \overline{g} \quad \textit{on } \mathbb{D}^2$$

for some holomorphic functions $f, g \in \mathcal{O}(\mathbb{D}^2)$. If $u \in L^2(\mathbb{T}^2)$, then $f, g \in H^2(\mathbb{D}^2)$ (in particular, $f, g \in L^2(\mathbb{T}^2)$). Moreover, after normalizing the decomposition by $g(0,0) = 0$, one has

$$\|u\|^2_{L^2(\mathbb{T}^2)} = \|f\|^2_{H^2(\mathbb{D}^2)} + \|g\|^2_{H^2(\mathbb{D}^2)},$$

and hence $\|f\|_{H^2(\mathbb{D}^2)},\ \|g\|_{H^2(\mathbb{D}^2)} \leq \|u\|_{L^2(\mathbb{T}^2)}$.

Proof. Let $c = g(0,0)$ and set $g_0 = g - c$ and $f_0 = f + \overline{c}$. Then $g_0(0,0) = 0$ and $u = f_0 + \overline{g_0}$. Replacing (f,g) by (f_0, g_0), we may assume $g(0,0) = 0$.

Write the Taylor expansions

$$f(z,w) = \sum_{m,n \geq 0} a_{m,n} z^m w^n, \qquad g(z,w) = \sum_{m,n \geq 0} b_{m,n} z^m w^n,$$

with $b_{0,0} = 0$. For $(\zeta, \eta) \in \mathbb{T}^2$ these yield the boundary Fourier series

$$f(\zeta,\eta) \sim \sum_{m,n \geq 0} a_{m,n}\, \zeta^m \eta^n, \qquad \overline{g}(\zeta,\eta) \sim \sum_{m,n \geq 0} \overline{b_{m,n}}\, \zeta^{-m} \eta^{-n}.$$

Since $\{\zeta^m \eta^n\}_{(m,n) \in \mathbb{Z}^2}$ is an orthonormal basis for $L^2(\mathbb{T}^2)$, the two functions f and $\overline{g}$ are orthogonal in $L^2(\mathbb{T}^2)$: indeed, any frequency (m,n) appearing in f has $m, n \geq 0$, while any frequency appearing in $\overline{g}$ has $m, n \leq 0$, and the only possible overlap would be the constant term $(0,0)$, which is absent from $\overline{g}$ because $b_{0,0} = 0$. Consequently,

$$\langle f, \overline{g} \rangle_{L^2(\mathbb{T}^2)} = 0.$$

Therefore, using $u = f + \overline{g} \in L^2(\mathbb{T}^2)$,

$$\|u\|^2_{L^2(\mathbb{T}^2)} = \|f + \overline{g}\|^2_{L^2} = \|f\|^2_{L^2} + \|\overline{g}\|^2_{L^2} = \|f\|^2_{L^2} + \|g\|^2_{L^2}.$$

In particular, $\|f\|_{L^2(\mathbb{T}^2)} < \infty$ and $\|g\|_{L^2(\mathbb{T}^2)} < \infty$. Since f and g are holomorphic on $\mathbb{D}^2$, this implies $f, g \in H^2(\mathbb{D}^2)$. Finally, by the standard coefficient characterization of $H^2(\mathbb{D}^2)$ (Parseval),

$$\|f\|^2_{H^2(\mathbb{D}^2)} = \sum_{m,n\geq 0} |a_{m,n}|^2, \qquad \|g\|^2_{H^2(\mathbb{D}^2)} = \sum_{m,n\geq 0} |b_{m,n}|^2,$$

and the displayed identity gives the asserted norm equality and estimates. □

3. Main results

Theorem 3.1. *Let $\Phi = (\varphi, \psi)$ be a holomorphic map from $\mathbb{D}$ to the bidisc $\mathbb{D}^2$, where $\varphi, \psi : \mathbb{D} \to \mathbb{D}$ are holomorphic self-maps. If*

$$\|\varphi\psi\|_\infty := \sup_{z\in\mathbb{D}} |\varphi(z)\psi(z)| < 1,$$

then the composition operator

$$C_\Phi : H^2_h(\mathbb{D}^2) \longrightarrow H^2_h(\mathbb{D}), \qquad (C_\Phi F)(z) = F(\varphi(z), \psi(z)),$$

is bounded.

We need the following holomorphic result, which is Theorem 2.1 in [1].

Proposition 3.1. *Under the assumptions of Theorem 3.1 the composition operator*

$$C_\Phi : H^2(\mathbb{D}^2) \longrightarrow H^2(\mathbb{D}), \qquad (C_\Phi F)(z) = F(\varphi(z), \psi(z)),$$

is bounded.

Proof. Choose σ with $\|\varphi\psi\|_\infty < \sigma < 1$ and set

$$\Gamma_1 := \{\zeta \in \partial\mathbb{D} : |\varphi(\zeta)| < \sqrt{\sigma}\}, \qquad \Gamma_2 := \partial\mathbb{D} \setminus \Gamma_1.$$

Then $m(\Gamma_1 \cup \Gamma_2) = 1$, and on Γ_2 we have $|\psi| < \sqrt{\sigma}$ a.e. because $|\varphi\psi| < \sigma$ a.e. on $\partial\mathbb{D}$.

For $F \in H^2(\mathbb{D}^2)$ write

$$F(z, w) = \sum_{n\geq 0} z^n F_n(w) = \sum_{n\geq 0} w^n G_n(z),$$

with $F_n, G_n \in H^2(\mathbb{D})$ and

$$\|F\|^2_{H^2(\mathbb{D}^2)} = \sum_{n\geq 0} \|F_n\|^2_{H^2(\mathbb{D})} = \sum_{n\geq 0} \|G_n\|^2_{H^2(\mathbb{D})}.$$

On Γ_1,

$$C_\Phi F = \sum_{n\geq 0} \varphi^n (C_\psi F_n),$$

so Cauchy–Schwarz and Tonelli give

$$\int_{\Gamma_1} |C_\Phi F|^2\, dm \leq \frac{1}{1-\sigma} \sum_{n\geq 0} \|C_\psi F_n\|^2_{H^2(\mathbb{D})}$$
$$\leq \frac{\|C_\psi\|^2}{1-\sigma} \sum_{n\geq 0} \|F_n\|^2_{H^2(\mathbb{D})} \leq \frac{\|C_\psi\|^2}{1-\sigma} \|F\|^2_{H^2(\mathbb{D}^2)}.$$

Similarly, on Γ_2,

$$C_\Phi F = \sum_{n\geq 0} \psi^n (C_\varphi G_n),$$

and hence

$$\int_{\Gamma_2} |C_\Phi F|^2\, dm \leq \frac{\|C_\varphi\|^2}{1-\sigma} \|F\|^2_{H^2(\mathbb{D}^2)}.$$

Adding the two estimates yields

$$\|C_\Phi F\|^2_{H^2(\mathbb{D})} \leq \frac{\|C_\psi\|^2 + \|C_\varphi\|^2}{1-\sigma} \|F\|^2_{H^2(\mathbb{D}^2)},$$

so C_Φ is bounded. □

Proof of Theorem 3.1. Let $u \in H^2_h(\mathbb{D}^2)$. By Proposition 2.1, we may decompose u as

$$u = f + \overline{g}$$

with f and g holomorphic on $\mathbb{D}^2$. Proposition 2.2 then shows that, after the harmless normalization $g(0,0) = 0$, both f and g actually belong to $H^2(\mathbb{D}^2)$ and satisfy

$$\|u\|^2_{H^2_h(\mathbb{D}^2)} = \|f\|^2_{H^2(\mathbb{D}^2)} + \|g\|^2_{H^2(\mathbb{D}^2)}.$$

This reduces the pluriharmonic problem to the corresponding holomorphic one.

Now compose with $\Phi = (\varphi, \psi)$. Since Φ is holomorphic, the holomorphic part remains holomorphic after composition, and the anti-holomorphic part remains anti-holomorphic. More precisely,

$$(C_\Phi u)(z) = u(\varphi(z), \psi(z)) = f(\varphi(z), \psi(z)) + \overline{g(\varphi(z), \psi(z))} = f \circ \Phi + \overline{g \circ \Phi}.$$

By Proposition 3.1, the composition operator $C_\Phi : H^2(\mathbb{D}^2) \to H^2(\mathbb{D})$ is bounded under the hypothesis $\|\varphi\psi\|_\infty < 1$. Hence

$$\|f \circ \Phi\|_{H^2(\mathbb{D})} \leq \|C_\Phi\|\, \|f\|_{H^2(\mathbb{D}^2)} \quad \text{and} \quad \|g \circ \Phi\|_{H^2(\mathbb{D})} \leq \|C_\Phi\|\, \|g\|_{H^2(\mathbb{D}^2)}.$$

Finally, in the one-variable pluriharmonic Hardy space $H^2_h(\mathbb{D})$, the holomorphic and anti-holomorphic parts are orthogonal after normalizing the constant term in the anti-holomorphic part. Therefore

$$\|C_\Phi u\|^2_{H^2_h(\mathbb{D})} = \|f \circ \Phi\|^2_{H^2(\mathbb{D})} + \|g \circ \Phi\|^2_{H^2(\mathbb{D})}.$$

Combining this identity with the previous bounds gives

$$\|C_\Phi u\|^2_{H^2_h(\mathbb{D})} \le \|C_\Phi\|^2\big(\|f\|^2_{H^2(\mathbb{D}^2)} + \|g\|^2_{H^2(\mathbb{D}^2)}\big) = \|C_\Phi\|^2\|u\|^2_{H^2_h(\mathbb{D}^2)}.$$

Taking square roots, we obtain

$$\|C_\Phi u\|_{H^2_h(\mathbb{D})} \le \|C_\Phi\|\, \|u\|_{H^2_h(\mathbb{D}^2)},$$

so C_Φ is bounded on $H^2_h(\mathbb{D}^2)$. □

Theorem 3.2. *Let $\Phi = (\varphi, \psi) : \mathbb{D} \to \mathbb{D}^2$ be a holomorphic map. Assume that*

$$\|\varphi\psi\|_\infty < 1 \quad \textit{and} \quad C_\varphi,\ C_\psi : H^2(\mathbb{D}) \to H^2(\mathbb{D}) \textit{ are compact.}$$

Then the composition operator

$$C_\Phi : H^2_h(\mathbb{D}^2) \to H^2_h(\mathbb{D}), \qquad (C_\Phi F)(z) = F(\varphi(z), \psi(z)),$$

is compact.

The proof relies on the following holomorphic theorem, namely Theorem 3.1 in [1].

Proposition 3.2. *Under the assumptions of Theorem 3.2 the composition operator*

$$C_\Phi : H^2(\mathbb{D}^2) \to H^2(\mathbb{D}), \qquad (C_\Phi F)(z) = F(\varphi(z), \psi(z)),$$

is compact.

Proof. Let $\{F_k\}$ be a bounded sequence in $H^2(\mathbb{D}^2)$ such that $F_k \to 0$ uniformly on compact subsets of $\mathbb{D}^2$. To prove compactness, it is enough to show that

$$\|C_\Phi F_k\|_{H^2(\mathbb{D})} \to 0.$$

Set

$$M := \sup_k \|F_k\|_{H^2(\mathbb{D}^2)} < \infty,$$

and choose a number σ satisfying $\|\varphi\psi\|_\infty < \sigma < 1$. As in the proof of Proposition 3.1, we decompose the boundary into

$$\Gamma_1 := \{\zeta \in \partial\mathbb{D} : |\varphi(\zeta)| < \sqrt{\sigma}\}, \qquad \Gamma_2 := \partial\mathbb{D} \setminus \Gamma_1.$$

On Γ_1 the factor φ is uniformly small, while on Γ_2 the hypothesis on $\varphi\psi$ forces ψ to be uniformly small almost everywhere. Thus it suffices to show that

$$\int_{\Gamma_j} |C_\Phi F_k|^2 \, dm \to 0 \qquad (j = 1, 2).$$

We treat the integral over Γ_1; the argument on Γ_2 is completely analogous after interchanging the roles of z and w, and of ψ and φ.

Write the z-expansion of F_k as

$$F_k(z, w) = \sum_{n \geq 0} z^n F_{n,k}(w),$$

where each $F_{n,k} \in H^2(\mathbb{D})$ and

$$\sum_{n \geq 0} \|F_{n,k}\|^2_{H^2(\mathbb{D})} = \|F_k\|^2_{H^2(\mathbb{D}^2)} \leq M^2.$$

In particular,

$$\|F_{n,k}\|_{H^2(\mathbb{D})} \leq M \qquad \text{for all } n, k.$$

For $\zeta \in \Gamma_1$ we have

$$(C_\Phi F_k)(\zeta) = F_k(\varphi(\zeta), \psi(\zeta)) = \sum_{n \geq 0} \varphi(\zeta)^n (F_{n,k} \circ \psi)(\zeta) = \sum_{n \geq 0} \varphi^n (C_\psi F_{n,k}).$$

Fix $N \in \mathbb{N}$. We split the above series into a finite head and an infinite tail:

$$C_\Phi F_k = \sum_{n=0}^{N} \varphi^n (C_\psi F_{n,k}) + \sum_{n > N} \varphi^n (C_\psi F_{n,k}).$$

The purpose of this decomposition is standard: compactness of C_ψ will control the finitely many low-order terms, while the smallness of $|\varphi|$ on Γ_1 will uniformly control the tail.

For the tail, since $|\varphi| < \sqrt{\sigma}$ on Γ_1, we obtain exactly as in the boundedness argument

$$\int_{\Gamma_1} \Big| \sum_{n > N} \varphi^n C_\psi F_{n,k} \Big|^2 dm \leq \sigma^{N+1} \|C_\Phi\|^2 M^2,$$

where we use the boundedness of $C_\Phi : H^2(\mathbb{D}^2) \to H^2(\mathbb{D})$ from Proposition 3.1. The key point is that the right-hand side is independent of k, so once N is chosen large, the tail is uniformly small for the whole sequence.

For the head, we estimate by Cauchy–Schwarz,

$$\int_{\Gamma_1} \Big| \sum_{n=0}^{N} \varphi^n C_\psi F_{n,k} \Big|^2 dm \leq (N+1) \sum_{n=0}^{N} \|C_\psi F_{n,k}\|^2_{H^2(\mathbb{D})}.$$

Now fix n. Since $F_k \to 0$ uniformly on compact subsets of $\mathbb{D}^2$, the coefficient functions $F_{n,k}$ also converge to 0 uniformly on compact subsets of $\mathbb{D}$. Because C_ψ is compact on $H^2(\mathbb{D})$ and the family $\{F_{n,k}\}_k$ is bounded in $H^2(\mathbb{D})$, it follows that

$$\|C_\psi F_{n,k}\|_{H^2(\mathbb{D})} \to 0 \qquad (k \to \infty).$$

Since only finitely many indices $0 \le n \le N$ occur in the head, we conclude that for each fixed N,

$$\int_{\Gamma_1} \Big| \sum_{n=0}^{N} \varphi^n C_\psi F_{n,k} \Big|^2 dm \to 0 \qquad (k \to \infty).$$

Combining the head and tail estimates gives the desired conclusion on Γ_1: given $\varepsilon > 0$, first choose N so that the tail is smaller than ε, and then choose k large enough so that the head is also smaller than ε. Therefore

$$\int_{\Gamma_1} |C_\Phi F_k|^2 \, dm \to 0.$$

The same reasoning on Γ_2, this time using the expansion in powers of w together with compactness of C_φ, yields

$$\int_{\Gamma_2} |C_\Phi F_k|^2 \, dm \to 0.$$

Adding the two pieces, we obtain

$$\|C_\Phi F_k\|^2_{H^2(\mathbb{D})} = \int_{\Gamma_1} |C_\Phi F_k|^2 \, dm + \int_{\Gamma_2} |C_\Phi F_k|^2 \, dm \to 0.$$

Hence C_Φ is compact. □

Proof of Theorem 3.2. To prove compactness on $H^2_h(\mathbb{D}^2)$, let $\{u_k\}$ be a bounded sequence in $H^2_h(\mathbb{D}^2)$ such that $u_k \to 0$ uniformly on compact subsets of $\mathbb{D}^2$. We must show that $\|C_\Phi u_k\|_{H^2_h(\mathbb{D})} \to 0$. For each k, Propositions 2.1 and 2.2 give a decomposition

$$u_k = f_k + \overline{g_k}, \qquad f_k, g_k \in H^2(\mathbb{D}^2),$$

which we normalize by requiring $g_k(0,0) = 0$. With this normalization, the norm identity from Proposition 2.2 yields

$$\|u_k\|^2_{H^2_h(\mathbb{D}^2)} = \|f_k\|^2_{H^2(\mathbb{D}^2)} + \|g_k\|^2_{H^2(\mathbb{D}^2)}.$$

Since the sequence $\{u_k\}$ is bounded, both $\{f_k\}$ and $\{g_k\}$ are bounded in $H^2(\mathbb{D}^2)$.

Moreover, because $u_k \to 0$ uniformly on compact subsets and the decomposition is obtained by separating holomorphic and anti-holomorphic parts, it follows that

$$f_k \to 0 \quad \text{and} \quad g_k \to 0 \qquad \text{uniformly on compact subsets of } \mathbb{D}^2.$$

Now compose with Φ. We have

$$C_\Phi u_k = u_k \circ \Phi = f_k \circ \Phi + \overline{g_k \circ \Phi}.$$

By Proposition 3.2, the holomorphic composition operator

$$C_\Phi : H^2(\mathbb{D}^2) \to H^2(\mathbb{D})$$

is compact under the hypotheses of the theorem. Applying this compactness first to the bounded sequence $\{f_k\}$ and then to $\{g_k\}$, we obtain

$$\|f_k \circ \Phi\|_{H^2(\mathbb{D})} \to 0, \qquad \|g_k \circ \Phi\|_{H^2(\mathbb{D})} \to 0.$$

Finally, in $H^2_h(\mathbb{D})$, the norm splits orthogonally into holomorphic and anti-holomorphic parts. Therefore

$$\|C_\Phi u_k\|^2_{H^2_h(\mathbb{D})} = \|f_k \circ \Phi\|^2_{H^2(\mathbb{D})} + \|g_k \circ \Phi\|^2_{H^2(\mathbb{D})}.$$

The right-hand side tends to 0, hence

$$\|C_\Phi u_k\|_{H^2_h(\mathbb{D})} \to 0.$$

This proves that $C_\Phi : H^2_h(\mathbb{D}^2) \to H^2_h(\mathbb{D})$ is compact. □

Note for contributor:

- Short bio: Nguyen Thi Minh Thao is affiliated with Vietnam Military Medical Academy, Vietnam. Nguyen Thu Huong is affiliated with Hanoi National University of Education, Vietnam.

- Author's contributions: Nguyen Thi Minh Thao: conceptualization, methodology, writing–original draft; Nguyen Thu Huong: supervision, validation, writing–review and editing.

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