HNUE JOURNAL OF SCIENCE
Natural Science, 2026, Volume 71, Issue 1, pp. 15-29
This paper is available online at http://hnuejs.edu.vn/ns
DOI: 10.18173/2354-1059.2026-0002

GLOBAL ATTRACTOR FOR A CLASS OF REACTION-DIFFUSION SYSTEMS

Tran Thi Quynh Chi
Department of Mathematics, Electric Power University, Hanoi, Vietnam
*Corresponding author: Tran Thi Quynh Chi, e-mail: chittq@epu.edu.vn

Received February 11, 2026. Revised March 20, 2026. Accepted March 30, 2026.

Abstract. We study the initial boundary value problem for a class of reaction-diffusion systems in bounded smooth domains, where one equation has a small diffusion coefficient $\delta > 0$, and also the corresponding limit system is formally obtained when $\delta = 0$. We prove the existence of global attractor $\mathcal{A}_\delta$ for the dynamical system generated by the system in both cases $\delta > 0$ and $\delta = 0$. Moreover, the upper semicontinuity of the global attractor $\mathcal{A}_\delta$ at $\delta = 0$ is also investigated.
Keywords: Reaction-diffusion system, limit system, Global attractor, upper-semicontinuity.

1. Introduction

In this paper, we investigate the long-time behavior of solutions to the following reaction-diffusion system in a bounded domain Ω in $\mathbb{R}^n$ with smooth boundary $\partial\Omega$:

$$\partial_t u = \Delta u - f(u, v) + g_1(x), \tag{1.1}$$

$$\partial_t v = \delta\Delta v - h(u, v) + g_2(x), \tag{1.2}$$

$$u|_{\partial\Omega} = 0, \qquad v|_{\partial\Omega} = 0, \tag{1.3}$$

$$u|_{t=0} = u_0 \in L_2(\Omega), \quad v|_{t=0} = v_0 \in L_2(\Omega). \tag{1.4}$$

where $\delta > 0$ is a small parameter, the nonlinear functions f, h and external forces g_1, g_2 satisfy certain conditions which will be specified.

Studying the asymptotic behavior of infinite-dimensional dynamical systems generated by nonlinear partial differential equations or functional differential equations has many practical applications. An approach to infinite-dimensional dissipative dynamical systems is to study the existence and properties of global attractors (see, e.g., the monographs [1], [2]). In recent years, the existence and properties of global attractors have been studied for various classes of nonlinear parabolic equations and systems appearing in heat transfer processes, diffusion processes, biology, and chemistry, etc, (see, e.g. [3]-[13]).

In this paper we study the existence and upper semicontinuity of global attractors for the dynamical system associated to problem (1.1)-(1.4). For this, we impose the following conditions:

$$f(u,v)u + h(u,v)v \geq \sigma(|u|^{p_1} + |v|^{p_2}) - C, \tag{1.5}$$

$$|f(u,v)|^{q_1} + |h(u,v)|^{q_2} \leq C_0(|u|^{p_1} + |v|^{p_2} + 1), \tag{1.6}$$

$$\left(f'_u.u + f'_v.v\right)u + \left(h'_u.u + h'_v.v\right)v \geq -C_1\left(|u|^2 + |v|^2\right), \quad u, v \in \mathbb{R}, \tag{1.7}$$

where σ, C, C_0, C_1, p_1 and p_2 are positive constants, $p_1, p_2 \geq 2$ and $q_i = p_i/(p_i - 1), i = 1, 2$. The functions $g_1(x)$ and $g_2(x)$ in equations (1.1) and (1.2) are assumed to satisfy the following conditions

$$g_1 \in L_2(\Omega), \quad g_2 \in L_2(\Omega). \tag{1.8}$$

The aim of this paper is to prove the existence of global attractors $\mathcal{A}_\delta$ in the case $\delta > 0$ and of the global attractor $\mathcal{A}_0$ in the case $\delta = 0$. We also prove the upper semicontinuity of $\{\mathcal{A}_\delta\}$ at $\delta = 0$. In the case $\delta > 0$, the associated dynamical system is strongly dissipative, thus the existence of a global attractor is not difficult to reach by using standard arguments. More precisely, due to the smoothing effect property, we can construct a bounded absorbing set in a suitable function space more regular than the phase space, so we can use the compactness of Sobolev embeddings to immediately ensure the asymptotic compactness of the associated dynamical system, and this implies the existence of a compact global attractor in the phase space. However, as in [7]-[11], [14], this smoothing effect is no longer holds in the case $\delta = 0$ since the second equation of the system is an ordinary differential equation, so the corresponding dynamical system is only weakly dissipative. Therefore, we have to overcome some essential difficulty when proving the asymptotic compactness in the limit case $\delta = 0$. To do this, we will utilize the ideas of splitting the associated semigroup into two parts as in the pioneering paper of Marion [14].

The remaining of this paper is organized as follows. In Section 2, we prove the existence of global attractors in the case $\delta > 0$. Section 3 is devoted to the existence of a global attractor in the limit case $\delta = 0$ and we give some examples for comparing our results with the existing literatureones of Marion [14]. Finally, in Section 4, we prove the upper semicontinuity of the family of global attractors $\{\mathcal{A}_\delta\}$ at $\delta = 0$.

2. Existence of global attractors in the case $\delta > 0$

In this section we prove the existence of global attractors for the dynamical system generated by problem (1.1)-(1.4) in the case $\delta > 0$. First, by the Galerkin method and compactness lemma, one can prove the following existence and uniqueness result.

Theorem 2.1. *[1] For given (u_0, v_0) in $L_2(\Omega) \times L_2(\Omega)$, under conditions* (1.5)-(1.8)*, there exists a unique global weak solution $(u(t), v(t))$ of the problem* (1.1)-(1.4) *such that*

$$u \in C\left([0,\infty); L_2(\Omega)\right) \cap L_2^{loc}\left(0,\infty; H_0^1(\Omega)\right) \cap L_{p_1}^{loc}\left(0,\infty; L_{p_1}(\Omega)\right),$$
$$v \in C\left([0,\infty); L_2(\Omega)\right) \cap L_2^{loc}\left(0,\infty; H_0^1(\Omega)\right) \cap L_{p_2}^{loc}\left(0,\infty; L_{p_2}(\Omega)\right),$$

and

$$\frac{du}{dt} \in L_{q_1}^{loc}\left(0,\infty; H^{-r_1}(\Omega)\right), \quad \frac{dv}{dt} \in L_{q_2}^{loc}\left(0,\infty; H^{-r_2}(\Omega)\right).$$

Moreover, the mapping $(u_0, v_0) \mapsto (u(t), v(t))$ is continuous in $(L_2(\Omega))^2$.

The above theorem allows us to construct a family of continuous semigroups

$$S_\delta(t) : L_2(\Omega) \times L_2(\Omega) \longrightarrow L_2(\Omega) \times L_2(\Omega)$$

associated with problem (1.1)-(1.4) as follows

$$S_\delta(t)(u_0, v_0) = (u(t), v(t)),$$

where $(u(t), v(t))$ is the unique global weak solution to problem (1.1)-(1.4) with initial datum (u_0, v_0).

Proposition 2.1. *The family of semigroups $\{S_\delta(t)\}_{\delta>0}$ associated to problem* (1.1)-(1.4) *has a bounded absorbing set in the space $(L_2(\Omega))^2$.*

Proof. By multiplying equations (1.1) and (1.2) with u and v, respectively, and integrating over Ω, we have

$$\frac{1}{2}\frac{d}{dt}\left(\|u(t)\|_{L_2}^2 + \|v(t)\|_{L_2}^2\right) + \|\nabla u(t)\|_{L_2}^2 + \delta\|\nabla u(t)\|_{L_2}^2$$
$$+ \int_\Omega \left[f(u(t),v(t))u(t) + h(u(t),v(t))v(t)\right] dx = \int_\Omega g_1 u(t) dx + \int_\Omega g_2 v(t) dx.$$

Using (1.5) and the elementary inequality $|a|^p \geq |a|^2 - 1$ for $p \geq 2$, we then obtain

$$\frac{1}{2}\frac{d}{dt}\left(\|u(t)\|_{L_2}^2 + \|v(t)\|_{L_2}^2\right) + \sigma\left(\|u(t)\|_{L_2}^2 + \|v(t)\|_{L_2}^2\right)$$
$$\leq (C + 2\sigma)|\Omega| + \frac{\sigma}{2}\left(\|u(t)\|_{L_2}^2 + \|v(t)\|_{L_2}^2\right) + \frac{1}{2\sigma}\left(\|g_1\|_{L_2}^2 + \|g_2\|_{L_2}^2\right).$$

Hence

$$\frac{d}{dt}\left(\|u(t)\|_{L_2}^2 + \|v(t)\|_{L_2}^2\right) + \sigma\left(\|u(t)\|_{L_2}^2 + \|v(t)\|_{L_2}^2\right) \leq C_2,$$

where $C_2 = 2(C+2\sigma)|\Omega| + \sigma^{-1}\left(\|g_1\|_{L_2}^2 + \|g_2\|_{L_2}^2\right)$. By multiplying the above inequality with $e^{\sigma t}$ and integrating in t, we get

$$\|u(t)\|_{L_2}^2 + \|v(t)\|_{L_2}^2 \leq \left(\|u(0)\|_{L_2}^2 + \|v(0)\|_{L_2}^2\right) e^{-\sigma t} + C_2\sigma^{-1}, \quad \forall t \geq 0. \tag{2.1}$$

This inequality implies that any ball in $(L_2(\Omega))^2$ centered at zero and of radius $\rho_2 > \rho_1 = \sqrt{C_2\sigma^{-1}}$ is an absorbing set in $(L_2(\Omega))^2$ for the semigroup $S_\delta(t)$. □

Proposition 2.2. *The family of semigroups* $\{S_\delta(t)\}_{\delta>0}$ *associated to problem* (1.1)-(1.4) *has an absorbing set in the space* $(H_0^1(\Omega))^2$.

Proof. Setting $\beta = \min\{1;\delta\}$, $\alpha = \beta\lambda_1$, where $\lambda_1 > 0$ is the first eigenvalue of the Laplace-Dirichlet operator $-\Delta_D$, we get from (1.5) and the Poincaré inequality that

$$\frac{d}{dt}(\|u\|_{L_2}^2 + \|v\|_{L_2}^2) + \beta\left(\|u\|_{H_0^1}^2 + \|v\|_{H_0^1}^2\right) \le 2C|\Omega| + \alpha^{-1}\left(\|g_1\|_{L_2}^2 + \|g_2\|_{L_2}^2\right). \quad (2.2)$$

Multiplying the above inequality with $e^{\alpha t}$ and integrating in t, we have

$$\begin{aligned} &\left(\|u(t)\|_{L_2}^2 + \|v(t)\|_{L_2}^2\right)e^{\alpha t} + \beta\int_0^t \left(\|u(s)\|_{H_0^1}^2 + \|v(s)\|_{H_0^1}^2\right)e^{\alpha s}ds \\ &\le \|u(0)\|_{L_2}^2 + \|v(0)\|_{L_2}^2 + 2C|\Omega|\alpha^{-1}e^{\alpha t} + \alpha^{-2}\left(\|g_1\|_{L_2}^2 + \|g_2\|_{L_2}^2\right)e^{\alpha t} \\ &\quad + \alpha\int_0^t \left(\|u(s)\|_{L_2}^2 + \|v(s)\|_{L_2}^2\right)e^{\alpha s}ds. \end{aligned} \quad (2.3)$$

Integrating (2.2) from 0 to t, we can deduce from the definitions of α and β that

$$\|u(t)\|_{L_2}^2 + \|v(t)\|_{L_2}^2 \le \left(\|u(0)\|_{L_2}^2 + \|v(0)\|_{L_2}^2\right)e^{-\alpha t} + R_1^2,$$

where $R_1^2 = 2C|\Omega|\alpha^{-1} + \alpha^{-2}\left(\|g_1\|_{L_2}^2 + \|g_2\|_{L_2}^2\right)$. Multiplying this inequality by $\alpha e^{\alpha t}$ and integrating in t, we get

$$\alpha\int_0^t \left(\|u(s)\|_{L_2}^2 + \|v(s)\|_{L_2}^2\right)e^{\alpha s}ds \le \alpha t\left(\|u(0)\|_{L_2}^2 + \|v(0)\|_{L_2}^2\right) + R_1^2 e^{\alpha t}.$$

We then obtain from (2.3) that

$$\int_0^t \left(\|u(s)\|_{H_0^1}^2 + \|v(s)\|_{H_0^1}^2\right)e^{\alpha s}ds \le \beta^{-1}(1+\alpha t)\left(\|u(0)\|_{L_2}^2 + \|v(0)\|_{L_2}^2\right) + R_2^2\beta^{-1}e^{\alpha t},$$

where $R_2^2 = 2C|\Omega|\alpha^{-1} + \alpha^{-2}\left(\|g_1\|_{L_2}^2 + \|g_2\|_{L_2}^2\right) + R_1^2$.

Now multiplying equations (1.1) and (1.2) with $-\Delta u$ and $-\Delta v$, respectively, and integrating over Ω, we have

$$\frac{1}{2}\frac{d}{dt}\left(\|u\|_{H_0^1}^2 + \|v\|_{H_0^1}^2\right) + \|\Delta u\|_{L_2}^2 + \delta\|\Delta v\|_{L_2}^2 - \int_\Omega [f(u,v)\Delta u + h(u,v)\Delta v]\,dx$$

$$= -\int_\Omega g_1\Delta u dx - \int_\Omega g_2\Delta v dx.$$

Using condition (1.7) and the Cauchy inequality, we obtain the following estimates

$$\int_\Omega [f(u,v)\Delta u + h(u,v)\Delta v]\,dx \le C_1\left(\|u\|_{H_0^1}^2 + \|v\|_{H_0^1}^2\right), \quad (2.4)$$

$$-\int_\Omega g_1 \Delta u dx \le \frac{1}{2}\|\Delta u\|_{L_2}^2 + \frac{1}{2}\|g_1\|_{L_2}^2, \tag{2.5}$$
$$-\int_\Omega g_2 \Delta v dx \le \frac{\delta}{2}\|\Delta v\|_{L_2}^2 + \frac{1}{2\delta}\|g_2\|_{L_2}^2.$$

From the above inequalities, we get

$$\frac{d}{dt}\left(\|u\|_{H_0^1}^2 + \|v\|_{H_0^1}^2\right) + \alpha\left(\|u\|_{H_0^1}^2 + \|v\|_{H_0^1}^2\right)$$
$$\le 2C_1\left(\|u\|_{H_0^1}^2 + \|v\|_{H_0^1}^2\right) + \|g_1\|_{L_2}^2 + \delta^{-1}\|g_2\|_{L_2}^2. \tag{2.6}$$

Multiplying (2.6) by $te^{\alpha t}$ and integrating in t, we have

$$t\left(\|u(t)\|_{H_0^1}^2 + \|v(t)\|_{H_0^1}^2\right)e^{\alpha t}$$
$$\le (2C_1 t + 1)\beta^{-1}\left[(1+\alpha t)\left(\|u(0)\|_{L_2}^2 + \|v(0)\|_{L_2}^2\right) + R_2^2 e^{\alpha t}\right]$$
$$+ t\left(\|g_1\|_{L_2}^2 + \delta^{-1}\|g_2\|_{L_2}^2\right)\alpha^{-1}e^{\alpha t}$$
$$\le (1+t+t^2)R_3^2\left(\|u(0)\|_{L_2}^2 + \|v(0)\|_{L_2}^2\right) + (1+t)R_4^2 e^{\alpha t}.$$

Finally, we get from the above estimates that

$$\|u(t)\|_{H_0^1}^2 + \|v(t)\|_{H_0^1}^2 \le (t+1+t^{-1})R_3^2\left(\|u(0)\|_{L_2}^2 + \|v(0)\|_{L_2}^2\right)e^{-\alpha t}$$
$$+(1+t^{-1})R_4^2, \quad \forall t \ge 0.$$

Hence, any ball $B_{H_0^1 \times H_0^1}(0, R_5)$, with $R_5 > R_4$, is an absorbing set for the semigroup $S_\delta(t)$ in $\left(H_0^1(\Omega)\right)^2$. □

Because the embedding $\left(H_0^1(\Omega)\right)^2 \hookrightarrow \left(L_2(\Omega)\right)^2$ is compact, we immediately get the following result.

Theorem 2.2. *Under assumptions* (1.5)-(1.8)*, the semigroup $S_\delta(t)$ has a compact global attractor in $\left(L_2(\Omega)\right)^2$.*

3. Existence of a global attractor in the case $\delta = 0$

In this section, we consider the limit system of equations (1.1)-(1.4) with diffusion coefficient $\delta = 0$:

$$\begin{cases} \partial_t u = \Delta u - f(u,v) + g_1(x), \\ \partial_t v = -h(u,v) + g_2(x), \\ u|_{\partial\Omega} = 0, \qquad v|_{\partial\Omega} = 0, \\ u|_{t=0} = u_0 \in L_2(\Omega), \quad v|_{t=0} = v_0 \in H_0^1(\Omega). \end{cases} \tag{3.1}$$

We keep the same notation as in the Introduction. In particular, the functions f, h satisfy conditions (1.5)-(1.7), and the functions g_1 and g_2 satisfy

$$g_1 \in L_2(\Omega), \quad g_2 \in H_0^1(\Omega). \tag{3.2}$$

We replace condition (1.6) by the two following conditions

$$|f(u,v)|^{q_1} \leq C_0 \left(|u|^{p_1} + |v|^{\min\{(p_2-1)q_1;p_2\}} + 1 \right), \tag{3.3}$$

$$|h(u,v)|^{q_2} \leq C_0 \left(|u|^{p_1} + |v|^{p_2} + 1 \right). \tag{3.4}$$

We also consider the following assumptions on the functions f and h:

$$h \in C^1(\mathbb{R}^2), \quad h(0,0) = 0, \tag{3.5}$$

$$\frac{\partial h}{\partial v}(u,v) \geq \sigma_1 > 0, \tag{3.6}$$

$$\left| \frac{\partial h}{\partial u}(u,v) \right| \leq D, \quad u, v \in \mathbb{R}, \tag{3.7}$$

and assume that the following inequality is satisfied

$$p_1 f(u,v) u |u|^{p_1-2} + p_2 h(u,v) v |v|^{p_2-2} \geq \sigma_2 \left(|u|^{2p_1-2} + |v|^{2p_2-2} \right) - C. \tag{3.8}$$

We note that the positive quantities σ_1, σ_2 can be arbitrarily small, so we can set $\sigma_1 = \sigma_2 = \sigma$.

Under assumptions (3.3)-(3.4), if $u \in L_{p_1}(0,T; L_{p_1}(\Omega)),\ v \in L_{p_2}(0,T; L_{p_2}(\Omega))$, then the functions f and h still satisfy with every $T > 0$,

$$f(u,v) \in L_{q_1}(0,T; L_{q_1}(\Omega)), \quad h(u,v) \in L_{q_2}(0,T; L_{q_2}(\Omega)).$$

Now, we can also show that problem (3.1) has a unique global weak solution by combining the Galerkin method and the compactness method. However, in this case, since the second equation of the system is an ordinary differential equation, we have additional difficulty in proving the existence of a global attractor. Although the technique is based on ideas in the pioneering paper [14] of Marion, the result for the existence of a global attractor as $\delta = 0$ seems to be new. We also give two examples in which the nonlinear terms do not satisfy the proposed conditions of Marion [14] but satisfy our conditions in the last section.

We have the following existence result.

Theorem 3.1. *Under assumptions* (1.5)-(1.7) *and* (3.2)-(3.8)*, problem* (3.1) *has a unique global weak solution* $(u(t), v(t))$ *satisfying*

$$u \in C\left([0,\infty); L_2(\Omega)\right) \cap L_2^{loc}\left(0,\infty; H_0^1(\Omega)\right) \cap L_{p_1}^{loc}\left(0,\infty; L_{p_1}(\Omega)\right),$$

$$v \in C\left([0,\infty); H_0^1(\Omega)\right) \cap L_\infty^{loc}\left(0,\infty; H_0^1(\Omega)\right) \cap L_{p_2}^{loc}\left(0,\infty; L_{p_2}(\Omega)\right),$$

and

$$\frac{du}{dt} \in L_{q_1}^{loc}\left(0,\infty; H^{-r_1}(\Omega)\right), \quad \frac{dv}{dt} \in L_{q_2}^{loc}\left(0,\infty; H^{-r_2}(\Omega)\right).$$

Moreover, the map $(u_0, v_0) \longmapsto (u(t), v(t))$ *is continuous in* $L_2(\Omega) \times H_0^1(\Omega)$.

The proof of Theorem 3.1 is similar to that of Theorem 2.1.

From Theorem 3.1, we can define a continuous semigroup $S_0(t) : L_2(\Omega) \times H_0^1(\Omega) \to L_2(\Omega) \times H_0^1(\Omega)$ associated with problem (3.1) by setting

$$S_0(t)(u_0, v_0) := (u(t), v(t)),$$

where $(u(t), v(t))$ is the unique weak solution of the problem (3.1) with initial datum (u_0, v_0).

We have the following theorem for the existence of a global attractor for the semigroup $S_0(t)$.

Theorem 3.2. *Let* $\{S_0(t)\}$ *be the semigroup associated with problem* (3.1). *Then, under assumptions* (1.5)-(1.7) *and* (3.2)-(3.8)*, the semigroup* $\{S_0(t)\}$ *possesses an* $(L_2 \times H_0^1, L_2 \times L_2)$*-global attractor* $\mathcal{A}_0$.

Proof. Similarly to the techniques used in Section 2, we can claim that any ball $B(0, \rho_2)$, with $\rho_2 > \rho_1 = \sqrt{C_2\sigma^{-1}}$, is an absorbing of the semigroup $S_0(t)$ in $(L_2(\Omega))^2$. Indeed, let $\mathcal{B}$ be a bounded set of $L_2(\Omega) \times H_0^1(\Omega)$, which is contained in $B(0, R)$, there exists $T_0 = \dfrac{1}{\sigma}\ln\dfrac{R^2}{\rho_2^2 - \rho_1^2}$ such that

$$S_0(t)\mathcal{B} \subset B(0, \rho_2), \quad \forall t \geq T_0.$$

We now prove the asymptotic compactness of this semigroup. To do this, we need the following lemma.

Lemma 3.1. *Let* $r > 0$ *be fixed. Then we have the following estimate*

$$\int_t^{t+r}\int_\Omega \left(|u|^{2p_1-2} + |v|^{2p_2-2}\right) dxds \leq K, \quad \forall t \geq 0,$$

where K *is a constant depending on initial datum* (u_0, v_0).

Proof. Let $(u_0, v_0) \in \mathcal{B} \subset B(0, R)$ and $t \geq T_0$, we have

$$\begin{aligned}\sigma\int_t^{t+r}\int_\Omega (|u|^{p_1} + |v|^{p_2})\, dxds &\leq C_3 r + \|u(t)\|_{L_2}^2 + \|v(t)\|_{L_2}^2 \\ &\leq C_3 r + \rho_2^2 \leq K, \quad \forall t \geq T_0, \end{aligned} \tag{3.9}$$

where $C_3 = 2(C+\sigma)|\Omega| + \sigma^{-1}\left(\|g_1\|_{L_2}^2 + \|g_2\|_{L_2}^2\right)$.

Multiplying the first and the second equations in (3.1) by $|u|^{p_1-2}u$ and $|v|^{p_2-2}v$, respectively, we get

$$\frac{d}{dt}\int_\Omega |u|^{p_1}dx - p_1\int_\Omega \Delta u.|u|^{p_1-2}udx + p_1\int_\Omega f(u,v)|u|^{p_1-2}udx$$
$$+\frac{d}{dt}\int_\Omega |v|^{p_2}dx + p_2\int_\Omega h(u,v)|v|^{p_2-2}vdx$$
$$= p_1\int_\Omega g_1|u|^{p_1-2}udx + p_2\int_\Omega g_2|v|^{p_2-2}vdx.$$

Since $-\int_\Omega \Delta u.|u|^{p_1-2}udx > 0$, we obtain

$$\frac{d}{dt}\int_\Omega \left(|u|^{p_1} + |v|^{p_2}\right)dx + p_1\int_\Omega f(u,v)|u|^{p_1-2}udx + p_2\int_\Omega h(u,v)|v|^{p_2-2}vdx$$
$$\leq p_1\int_\Omega g_1|u|^{p_1-2}udx + p_2\int_\Omega g_2|v|^{p_2-2}vdx.$$

Condition (3.8) implies that

$$\frac{d}{dt}\int_\Omega \left(|u|^{p_1} + |v|^{p_2}\right)dx + \frac{\sigma}{2}\int_\Omega \left(|u|^{2p_1-2} + |v|^{2p_2-2}\right)dx$$
$$\leq C|\Omega| + \frac{p_1^2}{2\sigma}\|g_1\|_H^2 + \frac{p_2^2}{2\sigma}\|g_2\|_H^2 \leq K. \tag{3.10}$$

Based on (3.9), by applying the uniform Gronwall inequality to (3.10), we get

$$\int_\Omega \left(|u(t)|^{p_1} + |v(t)|^{p_2}\right)dx \leq K, \quad \forall t \geq T_0 + r.$$

On the other hand, we obtain by integrating (3.10) in t that

$$\int_\Omega \left(|u(t+r)|^{p_1} + |v(t+r)|^{p_2}\right)dx - \int_\Omega \left(|u(t)|^{p_1} + |v(t)|^{p_2}\right)dx$$
$$+\frac{\sigma}{2}\int_t^{t+r}\int_\Omega \left(|u|^{2p_1-2} + |v|^{2p_2-2}\right)dxds \leq K.$$

Combining the two above inequalities, we have

$$\int_t^{t+r}\int_\Omega \left(|u|^{2p_1-2} + |v|^{2p_2-2}\right)dxds \leq K, \quad \forall t \geq T_0 + r.$$

The lemma is proved. □

Proof of Theorem 3.2. Let $\rho_2 > \rho_1$ and $r > 0$ be fixed. From the equation (2.1) and the assumptions (2.4)-(2.5), we can write the solution $v(t) = v_1(t) + v_2(t)$, with

$$\begin{cases} v_1(t) = & D\int_0^t u(s)e^{-\sigma(t-s)}ds + \int_0^t g_2 e^{-\sigma(t-s)}ds, \\ v_2(t) = & v(t) - v_1(t) \leq v(0)e^{-\sigma t}. \end{cases}$$

We define two families of operators from $L_2(\Omega) \times H_0^1(\Omega)$ into $L_2(\Omega) \times H_0^1(\Omega)$ by setting

$$S_1(t) : (u_0, v_0) \mapsto (u(t), v_1(t)),$$

$$S_2(t) : (u_0, v_0) \mapsto (0, v_2(t)).$$

For any bounded subset $\mathcal{B} \subset L_2(\Omega) \times H_0^1(\Omega)$, we have

$$r_{\mathcal{B}}(u_0, v_0) = \sup_{(u_0,v_0)\in\mathcal{B}} |S_2(t)(u_0, v_0)| \longrightarrow 0 \ \text{ as } \ t \to +\infty.$$

We can check that $S_1(t)$ is uniformly compact in the following sense: with every bounded set $\mathcal{B} \subset L_2(\Omega) \times H_0^1(\Omega)$, there exists a $t_0 \geq 0$ such that $\bigcup_{t\geq t_0} S_1(t)\mathcal{B}$ is relatively compact in $(L_2(\Omega))^2$.

We will use Lemma 3.1 in order to show that $S_1(t)$ is uniformly compact.

Estimate the component $u(t)$. Multiplying (2.1) by $-\Delta u$ and integrating over Ω, we obtain

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\|u\|_{H_0^1}^2 + \|\Delta u\|_{L_2}^2 &= \int_\Omega f(u,v)\Delta u dx - \int_\Omega g_2 \Delta u dx \\ &\leq \int_\Omega \left(|f(u,v)| + |g_2|\right)|\Delta u| dx. \end{aligned} \tag{3.11}$$

Using (3.3) we have

$$\begin{aligned} |f(u,v)| &\leq C_0^{1/q_1}\left(|u|^{p_1/q_1} + |v|^{\min\{p_2-1;p_2/q_1\}} + 1\right) \\ &\leq C_4\left(|u|^{p_1-1} + |v|^{p_2-1} + 1\right). \end{aligned}$$

The above inequality implies that

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\|u\|_{H_0^1}^2 + \|\Delta u\|_{L_2}^2 &\leq C_5\int_\Omega \left(|u|^{p_1-1} + |v|^{p_2-1} + 1 + |g_2|\right)|\Delta u| dx \\ &\leq \frac{1}{2}\|\Delta u\|_{L_2}^2 + \frac{C_5^2}{2}\int_\Omega \left(|u|^{p_1-1} + |v|^{p_2-1} + 1 + |g_2|\right)^2 dx. \end{aligned}$$

Hence,

$$\frac{d}{dt}\|u\|_{H_0^1}^2 \leq 4C_5^2\int_\Omega \left(|u|^{2p_1-2} + |v|^{2p_2-2} + 1 + |g_2|^2\right) dx. \tag{3.12}$$

We have the following estimates

$$\int_t^{t+r}\int_\Omega (1+|g_2|^2)dxds = r|\Omega| + r\|g_2\|_{L_2}^2,$$

$$\int_t^{t+r} \|u(s)\|_V^2 ds \le \frac{1}{2}(C_3 r + \rho_2^2), \quad \forall t \ge T_0.$$

Applying Lemma 3.1, there exists a constant $C_6 > 0$ such that

$$\int_t^{t+r}\int_\Omega \left(|u|^{2p_1-2} + |v|^{2p_2-2} + 1 + |g_2|^2\right) dxds \le C_6, \quad \forall t \ge T_0 + r.$$

Again by applying the uniformly Gronwall lemma to (3.12) we get

$$\|u(t)\|_{H_0^1}^2 \le C_7, \quad \forall t \ge T_0 + 2r, \tag{3.13}$$

with some $C_7 > 0$.

Estimate the component $v_1(t)$. With every $j = \overline{1,n}$, we set $w_j = \dfrac{\partial v_1}{\partial x_j}$; then w_j satisfies

$$\frac{\partial w_j}{\partial t} + \sigma w_j = D\frac{\partial u}{\partial x_j} + \frac{\partial g_2}{\partial x_j}.$$

Multiplying the above inequality by w_j and integrating over Ω, we get

$$\frac{d}{dt}\|w_j\|_{L_2}^2 + \sigma\|w_j\|_{L_2}^2 \le \frac{2D^2}{\sigma}\int_\Omega \left|\frac{\partial u}{\partial x_j}\right|^2 dx + \frac{2}{\sigma}\int_\Omega \left|\frac{\partial g_2}{\partial x_j}\right|^2 dx. \tag{3.14}$$

Summing (3.14) from $j = 1$ to $j = n$ and multiplying the received inequality by $e^{\sigma t}$, we get

$$\frac{d}{dt}\left(\|v_1\|_{H_0^1}^2 e^{\sigma t}\right) \le \frac{2D^2}{\sigma}\|u\|_{H_0^1}^2 e^{\sigma t} + \frac{2}{\sigma}\|g_2\|_{H_0^1}^2 e^{\sigma t}.$$

Again integrating the above inequality in t, we have

$$\begin{aligned}
\|v_1(t)\|_{H_0^1}^2 &\le \frac{2D^2}{\sigma}\int_0^t \|u(s)\|_{H_0^1}^2 e^{-\sigma(t-s)}ds + 2\sigma^{-2}\|g_2\|_{H_0^1}^2 \\
&\le \frac{2D^2}{\sigma}\int_0^{T_0+2r} \|u(s)\|_{H_0^1}^2 e^{-\sigma(t-s)}ds \\
&\quad + \frac{2D^2}{\sigma}\int_{T_0+2r}^t \|u(s)\|_{H_0^1}^2 e^{-\sigma(t-s)}ds + 2\sigma^{-2}\|g_2\|_{H_0^1}^2 \\
&\le \frac{D^2}{\sigma}[C_3(T_0+2r) + R^2] + \frac{2D^2}{\sigma^2}C_7 + 2\sigma^{-2}\|g_2\|_{H_0^1}^2, \quad \forall t \ge 0.
\end{aligned} \tag{3.15}$$

The estimates (3.13) and (3.15) provide the uniform compactness of the operator S_1. This shows that the semigroup $\{S_0(t)\}$ is asymptotically compact. Hence, the semigroup $S_0(t)$ has a global attractor $\mathcal{A}_0$ in the space $(L_2(\Omega))^2$. □

Now, we give two examples to show the differences between the results of Marion [14] and the results in this paper. In particular, the nonlinear terms in the following examples do not satisfy the conditions of Marion but satisfy our conditions.

Example 3.1. *Consider the following system*

$$\begin{cases} \dfrac{\partial u}{\partial t} = \Delta u - u^{p_1 - 1} - v + g_1(x), & \text{in } \Omega \times \mathbb{R}^+, \\ \dfrac{\partial v}{\partial t} = \dfrac{1}{2}u - \dfrac{p_1}{2}v + g_2(x), & \text{in } \Omega \times \mathbb{R}^+, \end{cases} \tag{3.16}$$

where $p_1 > 2$ *is an even number,* q_1 *conjugates with* p_1*,* $p_2 = q_2 = 2$*. The functions* g_1, g_2 *satisfy* $g_1 \in L_2(\Omega), g_2 \in H_0^1(\Omega)$*.*

To get the system of Marion, we define the functions f, h, σ and g by setting

$$\begin{cases} f(x,u,v) = v - g_1(x); \quad h(x,u) = u^{p_1 - 1}; \\ \sigma(x) = \dfrac{p_1}{2}; \quad g(x,u) = -\dfrac{1}{2}u - g_2(x); \quad p = p_1. \end{cases} \tag{3.17}$$

We also obtain the system (3.1) from setting

$$f(u,v) = u^{p_1 - 1} + v; \quad h(u,v) = -\frac{1}{2}u + \frac{p_1}{2}v. \tag{3.18}$$

By using some simple estimates, we can show that the above system satisfies the conditions in our paper. Condition (1.7) is satisfied with $C_1 = \dfrac{1}{4}$ and condition (3.8) is satisfied with $\sigma_2 = \dfrac{p_1 - 1}{2}$ and $C = \dfrac{1}{2}$. The remains are also checked easily.

However, if the function g_1 is not bounded in x, the condition for the boundedness of the function $f(x,u,v)$ in [14]:

$$|f(x,u,v)| \le \delta_4(1 + |u|^{p_1} + |v|), \quad 0 < p_1 < p - 1, \delta_4 > 0,$$

is not satisfied. On the other hand, if the function g_2 has unbounded partial derivatives in x_i, the following condition in [14] is not satisfied

$$|g'_u(x,u)| \le \delta_5, \quad |g'_{x_i}(x,u)| \le \delta_5(1 + |u|), \quad i = 1, \dots, n, \ \delta_5 > 0.$$

In the next example, we show that the conditions in [14] of Marion are not satisfied by the functions f, h belows.

Example 3.2. *Consider the following system*

$$\begin{cases} \dfrac{\partial u}{\partial t} = \Delta u - u^{p_1 - 1} - \dfrac{u^{p_1 - 1}v}{1 + v^2} + g_1(x), & \text{in } \Omega \times \mathbb{R}^+, \\ \dfrac{\partial v}{\partial t} = \dfrac{1}{2}u - v + g_2(x), & \text{in } \Omega \times \mathbb{R}^+, \end{cases} \tag{3.19}$$

where the constants p_1, q_1, p_2, q_2 *are assumed as in Example 3.1.*

We get the system (3.1) by setting

$$f(u,v) = u^{p_1-1} + \frac{u^{p_1-1}v}{1+v^2}; \quad h(u,v) = -\frac{1}{2}u + v. \tag{3.20}$$

The system of Marion is obtained by setting

$$\begin{cases} f(x,u,v) = \dfrac{u^{p_1-1}v}{1+v^2} - g_1(x); \quad h(x,u) = u^{p_1-1}; \\ \sigma(x) = 1; \quad g(x,u) = -\dfrac{1}{2}u - g_2(x); \quad p = p_1. \end{cases} \tag{3.21}$$

We easily check that the conditions (1.5)-(1.7) and (3.2)-(3.8) are satisfied for the above system. However, the function $f(x,u,v)$ does not satisfy the following condition of Marion by the exponent of $|u|$:

$$|f(x,u,v)| \leq \delta_4(1 + |u|^{p_1} + |v|), \quad 0 < p_1 < p-1, \delta_4 > 0.$$

Indeed,

$$|f(x,u,v)| = \left| \frac{u^{p-1}v}{1+v^2} - g_1(x) \right| \leq \frac{1}{2}|u|^{p-1} + |g_1(x)|.$$

4. Upper-semicontinuity of the global attractors $\mathcal{A}_\delta$ at $\delta = 0$

In the preceding sections, we have constructed the infinite-dimensional dynamical systems $(L_2(\Omega) \times H_0^1(\Omega), S_0(t))$ having a global attractor $\mathcal{A}_0$, and the dynamical system $(L_2(\Omega) \times L_2(\Omega), S_\delta(t))$ has global attractors $\mathcal{A}_\delta$. In this section, we will prove the upper semicontinuity of the global attractors $\mathcal{A}_\delta$ at $\delta = 0$. We need the following lemma.

Lemma 4.1. *[2] Assume that for $\delta \in [0, \delta_0)$, the semigroup $S_\delta(t)$ has a global attractor $\mathcal{A}_\delta$ and there exists a bounded set K such that*

$$\bigcup_{0 \leq \delta < \delta_0} \mathcal{A}_\delta \subset K. \tag{4.1}$$

If, in addition, the semigroup S_δ converges to S_0 in the sense that, for each $t > 0$, $S_\delta(t)(x,y) \to S_0(t)(x,y)$ uniformly on any bounded subset B of $L_2(\Omega) \times L_2(\Omega)$,

$$\sup_{(u_0,v_0) \in B} \|S_\delta(t)(u_0,v_0) - S_0(t)(u_0,v_0)\|_{L_2 \times L_2} \longrightarrow 0 \textit{ as } \delta \to 0^+, \tag{4.2}$$

then

$$\textit{dist}_{L_2 \times L_2}(\mathcal{A}_\delta, \mathcal{A}_0) \longrightarrow 0 \textit{ as } \delta \to 0^+.$$

Now, we are ready to prove the main theorem of this section.

Theorem 4.1. *The global attractor $\mathcal{A}_\delta$ of problem* (1.1)-(1.4) *converge as $\delta \to 0^+$ to the global attractor $\mathcal{A}_0$ of problem* (3.1)*, or the global attractors $\mathcal{A}_\delta$ is upper semicontinuous at $\delta = 0$ in the Hausdorff semi-distance*

$$dist_{L_2 \times L_2}(\mathcal{A}_\delta, \mathcal{A}_0) \longrightarrow 0 \text{ as } \delta \to 0^+.$$

Proof. We use Lemma 4.1 to show the upper semicontinuity of $\mathcal{A}_\delta$.
(i) Checking condition (4.1). We have showed in the preceding section that $B_\delta = B(0, \rho_2)$, with $\rho_2 > \sqrt{C_2 \sigma^{-1}}$, is an absorbing of (1.1)-(1.4) and (3.1) in the space $L_2(\Omega) \times L_2(\Omega)$.
For arbitrary $\delta_0 > 0$, with every $\delta \in [0, \delta_0)$, we have

$$\mathcal{A}_\delta = \bigcap_{s \geq 0} \overline{\bigcup_{t \geq s} S_\delta(t) B_\delta}^{L_2 \times L_2}.$$

Hence, for any with each $s \geq 0$,

$$\mathcal{A}_\delta \subset \overline{\bigcup_{t \geq s} S_\delta(t) B_\delta}^{L_2 \times L_2}.$$

We get $S_\delta(t) B_\delta \subset B_\delta$ with every $t \geq t_0$. This implies that

$$\mathcal{A}_\delta \subset \overline{B_\delta}^{L_2 \times L_2}, \ \forall \delta \in [0, \delta_0).$$

Therefore $\bigcup_{0 \leq \delta < \delta_0} \mathcal{A}_\delta \subset \overline{B_\delta}^{L_2 \times L_2} = \overline{B}(0, \rho_2)$. By setting $K = \overline{B}(0, \rho_2)$, the condition (4.1) is satisfied.
(ii) Checking condition (4.2). Assume that $(u_0, v_0) \in B \subset B(0, \rho)$ in $L_2(\Omega) \times L_2(\Omega)$. We have

$$\|S_\delta(t)(u_0, v_0) - S_0(t)(u_0, v_0)\|_{L_2 \times L_2} = \|\,(u_\delta(t) - u(t), v_\delta(t) - v(t))\,\|_{L_2 \times L_2},$$

where

$$(u_\delta(t), v_\delta(t)) \text{ is the solution of problem (1.1)} - \text{(1.4) with } \delta > 0,$$
$$(u(t), v(t)) \text{ is the solution of problem (3.1) with } \delta = 0.$$

Setting $u_1(t) = u_\delta(t) - u(t),\ v_1(t) = v_\delta(t) - v(t)$, we need to estimate

$$\sup_{(u_0, v_0) \in B} \|\,(u_1(t), v_1(t))\,\|_{L_2 \times L_2} \text{ as } \delta \to 0^+.$$

We have

$$\begin{cases} \partial_t u_1 = \Delta u_1 - [f(u_\delta, v_\delta) - f(u, v)], \\ \partial_t v_1 = \delta \Delta v_\delta - [h(u_\delta, v_\delta) - h(u, v)]. \end{cases}$$

Multiplying the above equations by u_1 and v_1, respectively, and integrating over Ω, we obtain

$$\frac{1}{2}\frac{d}{dt}\left(\|u_1\|_{L_2}^2+\|v_1\|_{L_2}^2\right)+\|\nabla u_1\|_{L_2}^2 \ =\delta\int_\Omega \Delta v_\delta(v_\delta-v)dx$$

$$-\int_\Omega \left[\left(f(u_\delta,v_\delta)-f(u,v)\right)(u_\delta-u)+\left(h(u_\delta,v_\delta)-h(u,v)\right)(v_\delta-v)\right]dx.$$

We have

$$\delta\int_\Omega \Delta v_\delta(v_\delta-v)dx\leq -\frac{\delta}{2}\|\nabla v_\delta\|_{L_2}^2+\ \frac{\delta}{2}\|v\|_{H_0^1}^2,$$

$$\int_\Omega \left[\left(f(u_\delta,v_\delta)-f(u,v)\right)(u_\delta-u)+\left(h(u_\delta,v_\delta)-h(u,v)\right)(v_\delta-v)\right]dx$$

$$\geq -C_1\left(\|u_1\|_{L_2}^2+\|v_1\|_{L_2}^2\right).$$

Hence, we get

$$\frac{d}{dt}\left(\|u_1\|_{L_2}^2+\|v_1\|_{L_2}^2\right)-2C_1\left(\|u_1\|_{L_2}^2+\|v_1\|_{L_2}^2\right)\leq \delta\|v\|_{H_0^1}^2.$$

We also have

$$\|v(t)\|_{H_0^1}^2\ \leq\ \|v_0\|_{H_0^1}^2e^{-\sigma t}+C_8\left(\|u_0\|_{L_2}^2+\|v_0\|_{L_2}^2\right)e^{-\sigma t}+C_9,\ \forall t\geq 0,$$

where $C_8=D^2\sigma^{-1},\ C_9=D^2C_2\sigma^{-2}+2\sigma^{-2}\|g_2\|_{H_0^1}^2$.

From $(u_0,v_0)\in B(0,\rho)$, there exists a positive constant C_{10} such that

$$\frac{d}{dt}\left(\|u_1\|_{L_2}^2+\|v_1\|_{L_2}^2\right)-2C_1\left(\|u_1\|_{L_2}^2+\|v_1\|_{L_2}^2\right)\leq \delta C_{10}.$$

Multiplying the above inequality by e^{-2C_1t} and integrating in t, we get

$$\|u_1(t)\|_{L_2}^2+\|v_1(t)\|_{L_2}^2\ \leq\left(\|u_1(0)\|_{L_2}^2+\|v_1(0)\|_{L_2}^2\right)e^{2C_1t}+\frac{\delta C_{10}}{2C_1}e^{2C_1t},\ \forall t\geq 0.$$

This implies that for any $t>0$

$$\|u_1(t)\|_{L_2}^2+\|v_1(t)\|_{L_2}^2\longrightarrow 0 \text{ as } \delta\to 0^+,$$

or

$$\sup_{(u_0,v_0)\in B}\|S_\delta(t)(u_0,v_0)-S_0(t)(u_0,v_0)\|_{H\times H}\longrightarrow 0 \text{ as } \delta\to 0^+.$$

The condition (4.2) is satisfied. Applying Lemma 4.1, we get the desired result. □

REFERENCES

[1] Chepyzhov VV & Vishik MI, 2002. *Attractors for Equations of Mathematical Physics*. American Mathematical Society, p. 588.

[2] Robinson JC, 2001. *Infinite-Dimensional Dynamical Systems.* Cambridge Texts in Applied Mathematics, Cambridge University Press, Cambridge, p. 480.

[3] Cung TA, Bui TH & Vu MT, 2026. Global attractor for a SEIRS model with diffusion and nonlinear incidence rates. *Journal of the Korean Mathematical Society*, 63(1), 145-181.

[4] Efendiev M & Miranville A, 2003. The dimension of the global attractor for dissipative reaction-diffusion systems. *Applied Mathematics Letters,* 16(3), 351-355.

[5] Kostianko A, Sun C & Zelik S, 2022. Reaction-diffusion systems with supercritical nonlinearities revisited. *Mathematische Annalen,* 384(1-2), 1-45.

[6] Lee J, Nguyen NT, & Pires L, 2023. Global attractors of generic reaction diffusion equations under Lipschitz perturbations. *Journal of Mathematical Analysis and Applications,* 528(2), Paper No. 127534, 22.

[7] Lee J & Vu MT, 2021. Global attractors and exponential stability of partly dissipative reaction diffusion systems with exponential growth nonlinearity. *Applicable Analysis,* 100(4), 735-751.

[8] Lu Y& Shao Z, 2003. Determining nodes for partly dissipative reaction diffusion systems. *Nonlinear Analysis,* 54(5), 873-884.

[9] Pu X & Zhang X, 2008. A remark on partly dissipative reaction diffusion systems on $\mathbb{R}^n$. *Acta Mathematicae Applicatae Sinica,* 24(4), 583-588.

[10] Rodriguez-Bernal A & Wang B, 2000. Attractors for partly dissipative reaction diffusion systems in $\mathbb{R}^n$. *Journal of Mathematical Analysis and Applications,* 252(2), 790-803.

[11] Shao Z, 2011. Existence and continuity of strong solutions of partly dissipative reaction diffusion systems. *Discrete and Continuous Dynamical Systems*, 31(4), 1319-1328.

[12] Mai XT & Vu MT, 2013. Long-time behavior for gradient parabolic systems in some unbounded domains. *Vietnam Journal of Mathematics*, 41(1), 97-113.

[13] Vishik MI & Chepyzhov VV, 2009. On trajectory attractors of reaction-diffusion systems with small diffusion. *Matematicheskii Sbornik*, 200(4), 471-497.

[14] Marion M, 1989. Finite-dimensional attractors associated with partly dissipative reaction-diffusion systems. *SIAM Journal on Mathematical Analysis,* 20(4), 816-844.