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COLLECTIVE EXCITATIONS OF A BOSE-EINSTEIN CONDENSATE TRAPPED IN A TWO-CHANNEL POTENTIAL

Nguyen Viet Hung
School of Materials Science and Engineering, Hanoi University of Science and Technology
Hanoi, Vietnam
Corresponding author: Nguyen Viet Hung, e-mail: hungvn1102@gmail.com

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Abstract. In this paper, we investigate the collective excitations of an attractively interacting Bose–Einstein condensate confined in a two-channel potential. Using the Bogoliubov approximation, we derive the Bogoliubov–de Gennes equations and determine the excitation spectrum numerically. Two distinct modes are identified: the well-known Josephson mode, associated with periodic inter-channel tunneling, and a novel antisymmetric translational mode, describing out-of-phase oscillations of the condensate center of mass. Unlike conventional dipole oscillations, the latter arises from the interplay between inter-channel coupling and attractive nonlinearity and does not follow the Kohn theorem. We then employ the variational approximation to describe the condensate dynamics under these excitations and to extract the corresponding mode frequencies. The results are further validated by real-time simulations of the governing Gross–Pitaevskii equations, showing good agreement between analytical and numerical approaches.
Keywords: Collective excitations, Gross-Pitaevskii equation, Bogoliubov - de Gennes equation, Variational approximation.

1. Introduction

BoseEinstein condensation is a quantum phase transition that occurs when an ideal gas of bosons is cooled below a critical temperature. This phenomenon was predicted in 1924-1925 by Satyendra Nath Bose and Albert Einstein [1], and was first experimentally observed in 1995 in dilute alkali gases, which are systems of weakly interacting bosons [2]-[3]. Under these conditions, many particles occupy the same quantum ground state and behave coherently as a single entity [4], [5]. The system can be described by a macroscopic wave function. BoseEinstein condensates (BEC) provide a unique setting to explore quantum phenomena on a macroscopic scale, such as superfluidity, quantum phase transitions, nonlinear matter-wave dynamics, and other collective quantum effects [6]-[10]. In the last two decades, the field of BoseEinstein condensation has expanded into condensed matter physics, where the condensation of quasiparticles such as excitonpolaritons has attracted considerable attention [11], [12].

Many intriguing collective quantum effects have been studied in BECs under various physical conditions. Collective excitations are a fundamental aspect of BECs, serving as a sensitive probe of the underlying interparticle interactions. These excitations represent the coordinated motion of a large number of constituent particles within the condensate, offering crucial insights into the dynamics and fundamental properties of these quantum systems [5]. In the experiment, the collective modes were excited by applying a weak time-dependent perturbation at a specific frequency to the transverse component of the trapping potential. The resulting real-time dynamics, observed as shape oscillations of the condensate, were then measured. From these observations, low-lying eigenmodes with different symmetries were identified [13]. At the mean-field level, these modes can theoretically be well described by the time-dependent GrossPitaevskii equation that governs the macroscopic wave function of a BEC [4].

Collective excitations have been intensively investigated in dilute-gas BEC via high-partial-wave Feshbach resonances [14], in spinorbit-coupled BECs [15], in dipolar BECs and quantum droplets [16], [17], in low-dimensional BECs with quantum fluctuations [18], and in bound-in-the-continuum condensates [19]. The second-harmonic generation of propagating collective excitations in a two-component BEC was studied in [20].

Among the many studies of Bose–Einstein condensates under various physical conditions [13]-[20], those confined in dual-core potentials have attracted considerable attention due to their rich physics, including Josephson oscillations, symmetry breaking in soliton formation, and anomalous Bogoliubov modes [9, 21, 22]. In this work, we consider a Bose–Einstein condensate trapped in a dual-core potential with sufficiently tight confinement in each core, such that the dynamics occur primarily within individual cores, while the coupling between them arises solely from weak tunneling. The system can thus be effectively described by a two-channel model, within which the collective excitations of the condensate are investigated.

The paper is organized as follows. In Section 2, we introduce the physical model and the Bogoliubov method used for the numerical analysis of BEC excitations. Section 3 is devoted to the variational approximation (VA), an analytical method for exploring the characteristics of the collective modes and comparing them with numerical results. Section 4 summarizes the main findings of our study.

2. The model and Bogoliubov approximation

We consider a Bose–Einstein condensate trapped in an external magnetic-field potential forming a double-channel structure, as depicted in Fig. 1. Here, we assume that the confinement along the $\tilde{y}$ direction is sufficiently strong, such that the atomic motion in this direction is effectively frozen, with only weak tunneling between the channels. Consequently, the dynamics take place primarily along the $\tilde{x}$ direction. Within the mean-field approximation, the quantum dynamics of the condensate in such a potential is governed by the following coupled GrossPitaevskii equations [9]

$$i\hbar\frac{\partial\Psi_1(\tilde{x},\tilde{t})}{\partial\tilde{t}} = -\frac{\hbar^2}{2m}\frac{\partial^2\Psi_1(\tilde{x},\tilde{t})}{\partial\tilde{x}^2} - g|\Psi_1(\tilde{x},\tilde{t})|^2\Psi_1(\tilde{x},\tilde{t}) - \kappa\Psi_2(\tilde{x},\tilde{t}), \tag{2.1}$$

$$i\hbar\frac{\partial\Psi_2(\tilde{x},\tilde{t})}{\partial\tilde{t}} = -\frac{\hbar^2}{2m}\frac{\partial^2\Psi_2(\tilde{x},\tilde{t})}{\partial\tilde{x}^2} - g|\Psi_2(\tilde{x},\tilde{t})|^2\Psi_2(\tilde{x},\tilde{t}) - \kappa\Psi_1(\tilde{x},\tilde{t}), \tag{2.2}$$

where m is the mass of an individual atom in the condensate, g exhibits nonlinear interaction between atoms, and κ identifies the linear tunneling of atoms between potential channels. The functions $\Psi_1(\tilde{x},\tilde{t})$ and $\Psi_2(\tilde{x},\tilde{t})$ are the wave functions of the Bose–Einstein condensate in the two channels, while $\tilde{x}$ and $\tilde{t}$ denote the spatial and temporal variables in physical units. The total number of atoms in the condensate is

$$N_{atoms} = \int_{-\infty}^{\infty}\left(|\Psi_1(\tilde{x},\tilde{t})|^2 + |\Psi_2(\tilde{x},\tilde{t})|^2\right) d\tilde{x}. \tag{2.3}$$

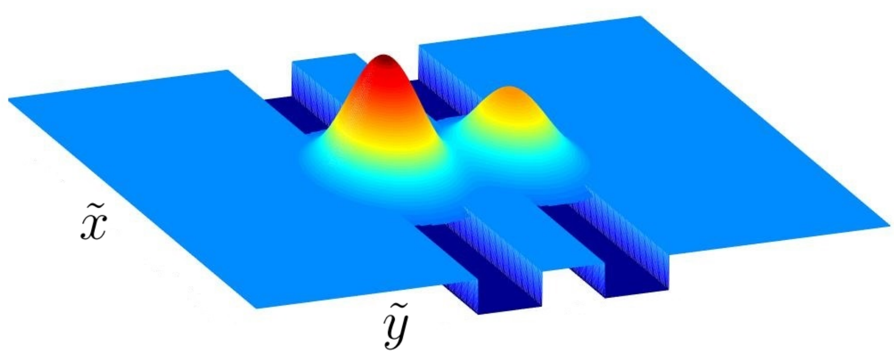

Figure 1. Schematic illustration of a Bose-Einstein condensate confined within a two-channel potential

By introducing the scaled variables $x = \tilde{x}\sqrt{m\kappa}/\hbar$, $t = \tilde{t}\kappa/\hbar$ and $\psi_j = \Psi_j\sqrt{g/\kappa}$, Eqs. (2.1) and (2.2) can be cast into a dimensionless form.

$$i\frac{\partial\psi_1}{\partial t} = -\frac{1}{2}\frac{\partial^2\psi_1}{\partial x^2} - |\psi_1|^2\psi_1 - \psi_2, \tag{2.4}$$

$$i\frac{\partial\psi_2}{\partial t} = -\frac{1}{2}\frac{\partial^2\psi_2}{\partial x^2} - |\psi_2|^2\psi_2 - \psi_1. \tag{2.5}$$

The total number of atoms (2.3) in the new variables is now given by

$$N = \int_{-\infty}^{\infty} \left(|\psi_1(x,t)|^2 + |\psi_2(x,t)|^2\right) dx = \frac{g}{\hbar}\sqrt{\frac{m}{\kappa}} N_{atoms}. \tag{2.6}$$

Notice that after rescaling, the system (2.4)-(2.5) is characterized by just one dimensionless parameter N. It is called the norm of the condensate wavefunction [4]-[7]. The norm N is conserved in the time evolution of BEC as the number of atoms N_{atoms} in the system is unchanged.

In the case of dilute condensates, corresponding to small values of the norm N, Eqs. (2.4) - (2.5) support symmetric bright-soliton solutions [9, 23]

$$\psi_1(x,t) = \psi_2(x,t) = \psi_S(x)e^{-i\mu t} = \frac{N}{4}\,\text{sech}\left(\frac{Nx}{4}\right)e^{-i\mu t} \tag{2.7}$$

where $\mu = -\left(1 + \frac{N^2}{32}\right)$. In order to determine the collective excitations of the condensate, we apply here the Bogoliubov approximation [24]. The excited wavefunction of the condensate can be represented in the following way

$$\psi_1(x,t) = \left(\psi_S(x) + \delta_1(x,t)\right)e^{-i\mu t}, \tag{2.8}$$

$$\psi_2(x,t) = \left(\psi_S(x) + \delta_2(x,t)\right)e^{-i\mu t}, \tag{2.9}$$

where the perturbations $\delta_1(x,t)$ and $\delta_2(x,t)$ are small compared to ψ_S. Substituting these expressions into Eqs. (2.4) and (2.5), we expand all terms to first order in the perturbations $\delta_1(x,t)$ and $\delta_2(x,t)$, while systematically neglecting higher-order contributions. After straightforward algebra and using the stationary equation satisfied by $\psi_S(x)$ to eliminate the zeroth-order terms, we collect the linear contributions and arrive at the coupled Bogoliubov–de Gennes equations

$$i\frac{\partial\delta_1}{\partial t} = -\frac{1}{2}\frac{\partial^2\delta_1}{\partial x^2} + \left(2i\psi_S(x)^2 - \mu\right)\delta_1 - \psi_S(x)^2\delta_1^* - \delta_2, \tag{2.10}$$

$$i\frac{\partial\delta_2}{\partial t} = -\frac{1}{2}\frac{\partial^2\delta_2}{\partial x^2} + \left(2i\psi_S(x)^2 - \mu\right)\delta_2 - \psi_S(x)^2\delta_2^* - \delta_1. \tag{2.11}$$

To decouple these equations, we introduce two new functions

$$\delta_S(x,t) = \delta_1(x,t) + \delta_2(x,t). \tag{2.12}$$

$$\delta_D(x,t) = \delta_1(x,t) - \delta_2(x,t). \tag{2.13}$$

Consequently, the equations (2.10)-(2.11) split into two separate equations

$$i\frac{\partial \delta_S}{\partial t} = -\frac{1}{2}\frac{\partial^2 \delta_S}{\partial x^2} + 2i\psi_S(x)^2\delta_S - \psi_S(x)^2\delta_S^* - (\mu+1)\delta_S, \tag{2.14}$$

$$i\frac{\partial \delta_D}{\partial t} = -\frac{1}{2}\frac{\partial^2 \delta_D}{\partial x^2} + 2i\psi_S(x)^2\delta_D - \psi_S(x)^2\delta_D^* - (\mu-1)\delta_D. \tag{2.15}$$

The function $\delta_S(x,t)$ can be expanded in terms of excited modes of the equation (2.14) as follows

$$\begin{pmatrix}\delta_S\\ \delta_S^*\end{pmatrix} = \sum_{n=1}^{\infty}\left(\alpha_n e^{-i\Omega_n t}\begin{pmatrix}u_n\\ v_n\end{pmatrix} + \alpha_n^* e^{i\Omega_n t}\begin{pmatrix}v_n^*\\ u_n^*\end{pmatrix}\right), \tag{2.16}$$

and similar for the function $\delta_D(x,t)$. Here Ω_n is the mode frequency, while $u_n(x)$ and $v_n(x)$ are the mode shape functions, and α_n is the corresponding amplitude. To analyze the mode structure (2.16) of the Bogoliubovde Gennes equations (2.14)(2.15), we use the finite-difference method to discretize them into matrix form and then perform diagonalization [25]. The collective excited modes of the BoseEinstein condensate correspond to the localized modes of Eqs. (2.14) and (2.15).

The numerical results clearly show that Eq. (2.14) admits no localized modes, whereas Eq. (2.15) admits two. In Figure 2, the moduli of the mode shape functions $\{u(x), v(x)\}$ of the two localized modes of Eq. (2.15) are shown as a typical example. Here $N = 2.6$ and the mode frequencies are $\Omega_1 = 1.66$ and $\Omega_2 = 2.14$. The dependence of the mode frequencies on N will be presented in the next section (Figs. 5 and 6), where we compare numerical results with the analytical approximation.

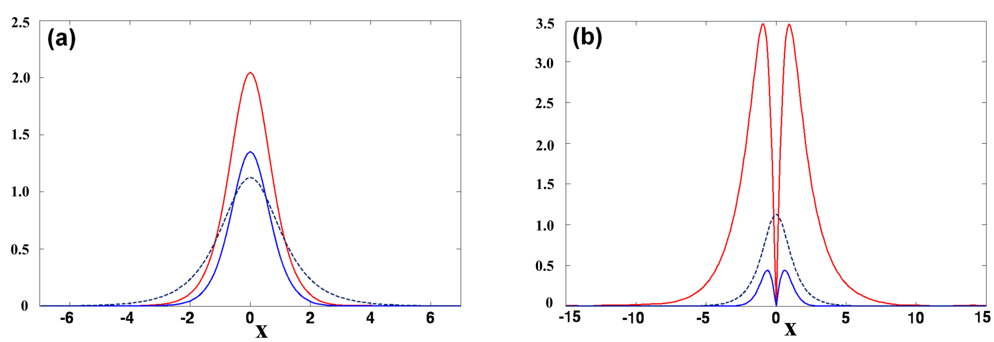

Figure 2. The first (a) and the second (b) collective excitation modes of the BoseEinstein condensate with $N = 2.6$. The dashed curves show the moduli of the condensate wave functions $\psi_S(x)$. The red and blue curves indicate the moduli of the mode functions $u(x)$ and $v(x)$, respectively

By examining the mode functions $\{u(x),\, v(x)\}$ shown in Figure 2, one can gain intuitive insight into the physical nature of the excitations. For the first mode, the pair $\{u(x),\, v(x)\}$ exhibits symmetric spatial profiles with respect to the center. When this mode is superimposed onto the stationary condensate, it primarily induces a modulation of the density amplitude over time, without significantly altering the spatial position of the wavefunction. As a result, the perturbation leads to oscillations of the population imbalance between the two channels, corresponding to a periodic transfer of atoms while preserving the total norm. This behavior is characteristic of the Josephson mode. In contrast, the functions $\{u(x),\, v(x)\}$ of the second mode are antisymmetric and vanish at the center, implying opposite perturbations in the two channels. Exciting this mode induces out-of-phase displacements of the condensate components while preserving their shapes. The dynamics is therefore governed by the relative translational degree of freedom, identifying this excitation as an antisymmetric translational mode.

These features can be clearly seen in Figs. 3 and 4, where the first and second modes are added to the condensate at the initial time with small amplitudes (as perturbations), and the subsequent quantum dynamics is observed. Here, we employ the split-step Fourier method for coupled nonlinear partial differential equations, as developed in previous works [27]-[30], to perform real-time simulations of the coupled Gross-Pitaevskii equations (2.4)-(2.5).

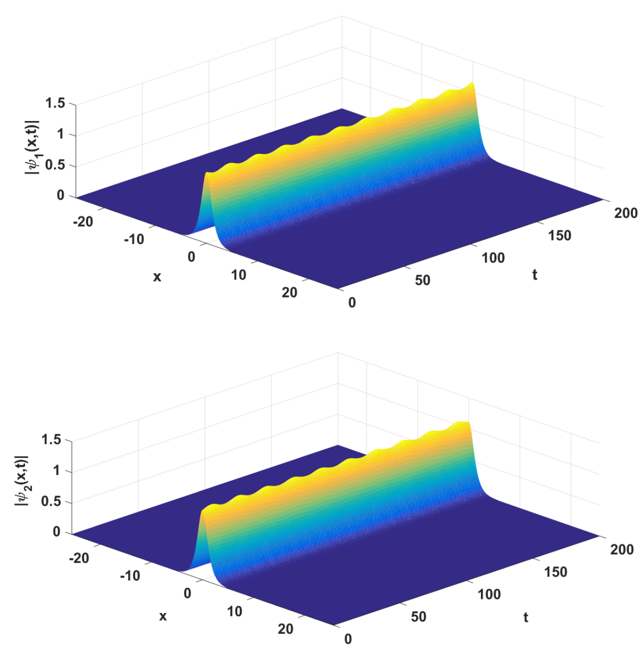

Figure 3. Evolution of the Bose-Einstein condensate under excitation of the first collective mode

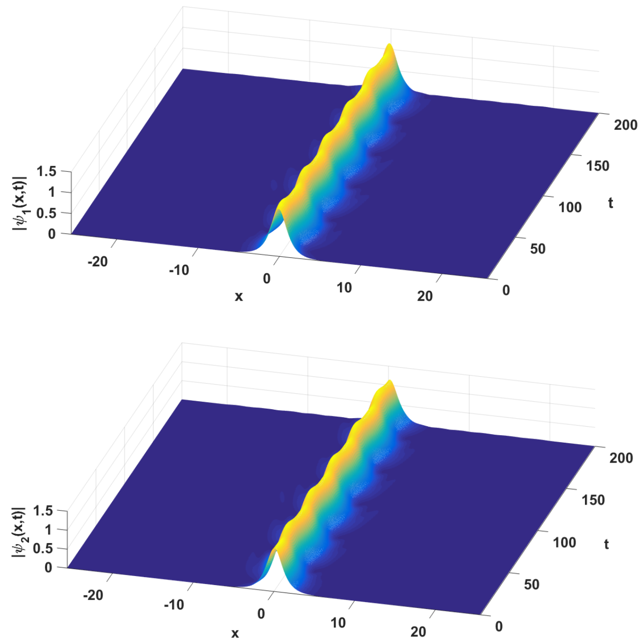

Figure 4. Evolution of Bose-Einstein condensate under second collective mode excitation

3. Variational Approximation

To gain further insight into the numerical results presented in the previous section, we apply the Variational Approximation (VA). The VA is a powerful non-perturbative method that enables us to capture key features of the dynamics of complex nonlinear physical systems [8], [23], [26].

Within the dynamical variational approach [8]-[9], the Lagrangian describing the equations (2.4)-(2.5) reads

$$\begin{aligned} L = \int_{-\infty}^{\infty} \Bigg[\frac{1}{2} \sum_{j=1}^{2} i \left(\psi_j \frac{\partial \psi_j^*}{\partial t} - \psi_j^* \frac{\partial \psi_j}{\partial t} \right) \\ - \frac{1}{2} \sum_{j=1}^{2} \left| \frac{\partial \psi_j}{\partial x} \right|^2 + \frac{1}{2} \sum_{j=1}^{2} |\psi_j|^4 + (\psi_1 \psi_2^* + \psi_2 \psi_1^*) \Bigg] dx \end{aligned} \tag{3.1}$$

It has been shown in the previous section that the first excited collective mode corresponds to the periodic tunneling of condensed atoms between two channels. To characterize such behavior, one can introduce the usual parameter

$$\nu = \frac{1}{N} \left(\int_{-\infty}^{\infty} |\psi_1|^2 \, dx - \int_{-\infty}^{\infty} |\psi_2|^2 \, dx \right) \tag{3.2}$$

that measures the imbalance of norms in the two channels. Therefore, the adopted ansatz for the condensate, used to determine the first collective mode, is taken as

$$\psi_1(x,t) = \sqrt{\frac{N(1+\nu)}{4W}}\,\operatorname{sech}\!\Big(\frac{x}{W}\Big)\exp\left[-i\left(\frac{bx^2}{2}+\phi_1\right)\right], \tag{3.3}$$

$$\psi_2(x,t) = \sqrt{\frac{N(1-\nu)}{4W}}\,\operatorname{sech}\!\Big(\frac{x}{W}\Big)\exp\left[-i\left(\frac{bx^2}{2}+\phi_2\right)\right], \tag{3.4}$$

where $\nu(t)$, $W(t)$, $\phi_1(t)$, $\phi_2(t)$, and $b(t)$ are time-dependent variational parameters, representing the population imbalance, soliton width, phases in each channel, and chirp, respectively.

Substituting the ansatz (3.3)–(3.4) into the Lagrangian (3.1) and performing the integration over x, we obtain

$$L = \frac{N}{24W^2}\Big[2NW(1+\nu^2) - 4 - \pi^2 W^4(b' + b^2) + 12W^2\left(2\sqrt{1-\nu^2}\cos(\Delta\phi) - \nu\Delta\phi' - \phi'\right)\Big] \tag{3.5}$$

where $\Delta\phi = \phi_1 - \phi_2$ and $\phi = \phi_1 + \phi_2$. The equations of motion for the variational parameters are obtained from the Euler-Lagrange equations

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = 0, \tag{3.6}$$

with $q_i \in \{W, b, \nu, \Delta\phi, \phi\}$. Applying this procedure to the reduced Lagrangian (3.5), we arrive at

$$b' = \frac{4}{\pi^2 W^4} - \frac{N(1+\nu^2)}{\pi^2 W^3} - b^2, \tag{3.7}$$

$$W' = bW, \tag{3.8}$$

$$\nu' = 2\sin(\Delta\phi)\sqrt{1-\nu^2}, \tag{3.9}$$

$$\Delta\phi' = \frac{N\nu}{3W} - \frac{2\nu\cos(\Delta\phi)}{\sqrt{1-\nu^2}}. \tag{3.10}$$

The equation for ϕ is omitted, as it decouples from the above system and only reflects the conservation of the total norm.

We eliminate the parameter b from the above system of equations and obtain the following system in more compact form

$$W'' = \frac{4}{\pi^2 W^3} - \frac{N(1+\nu^2)}{\pi^2 W^2}, \tag{3.11}$$

$$\nu' = 2\sin(\Delta\phi)\sqrt{1-\nu^2}, \tag{3.12}$$

$$\Delta\phi' = \frac{N\nu}{3W} - \frac{2\nu\cos(\Delta\phi)}{\sqrt{1-\nu^2}}. \tag{3.13}$$

It is worth noting that the variations of the parameters $W(t)$, $\nu(t)$ and $\Delta\phi(t)$ during the evolution are small compared to their equilibrium values. Therefore, we can linearize Eqs. (3.11)-(3.13) to obtain the following equations

$$\delta W'' = -\frac{N^4}{64\pi^2}\,\delta W, \tag{3.14}$$

$$\nu'' = -4\left(1 - \frac{N^2}{24}\right)\nu, \tag{3.15}$$

$$\Delta\phi'' = -4\left(1 - \frac{N^2}{24}\right)\Delta\phi. \tag{3.16}$$

Here we observe harmonic oscillations of characteristic parameters. The important quantity is the frequency of oscillation of the norm imbalance parameter $\nu(t)$ in (3.15), which is given by

$$\Omega_1 = 2\sqrt{1 - \frac{N^2}{24}}. \tag{3.17}$$

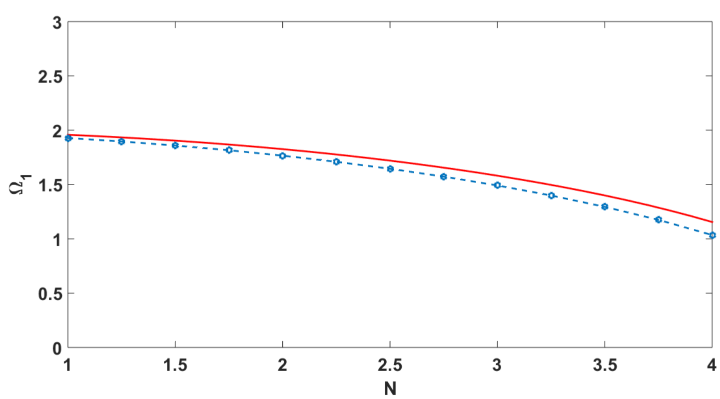

Figure 5. The frequency of the first collective mode of Bose-Einstein condensate as a function of norm N. The red curve is the prediction of variational approximation (3.17) while the dots are results from diagonalization of Bogoliubov-de Gennes equation (2.15)

Obviously, the oscillation of $\nu(t)$ is directly related to the periodic tunneling of condensed atoms between the two channels, as illustrated in Figure 3. A comparison between the numerical results obtained using the Bogoliubov method and the predictions of the VA is presented in Figure 5. We conclude that the two results are in good agreement.

In order to derive an explicit formula for the frequency of the second excited mode, we need to choose the appropriate ansatz which refers to the oscillatory motion of both condensate components within the channels about the center of mass, as can be seen in Figure 4. Therefore, we propose the following ansatz

$$\psi_1(x,t) = \sqrt{\frac{N}{W\sqrt{2\pi}}}\exp\left[-\frac{(x-b)^2}{W^2}\right]\exp\Big[-i\,(cx+\varphi_1)\Big], \tag{3.18}$$

$$\psi_2(x,t) = \sqrt{\frac{N}{W\sqrt{2\pi}}}\exp\left[-\frac{(x+b)^2}{W^2}\right]\exp\Big[-i\,(\varphi_2-cx)\Big]. \tag{3.19}$$

where $b(t)$, $W(t)$, $\varphi_1(t)$, $\varphi_2(t)$, and $c(t)$ are variational parameters. The magnitudes of the parameters $b(t)$ and $c(t)$ are assumed to be very small.
In this case, the Lagrangian (3.1) becomes

$$L = N\left[\frac{NW-2\sqrt{\pi}}{4\sqrt{\pi}W^2} - \frac{c^2}{2} + \frac{\varphi'}{2} + bc' - \cos(\Delta\varphi)\,\exp\left(-\frac{c^2W^2}{2} - \frac{2b^2}{W^2}\right)\right] \tag{3.20}$$

where $\Delta\varphi = \varphi_1 - \varphi_2$ and $\varphi = \varphi_1 + \varphi_2$. The EulerLagrange equations for the variational parameters are

$$W^2c' - 4b\cos(\Delta\varphi)\,\exp\left(-\frac{c^2W^2}{2} - \frac{2b^2}{W^2}\right) = 0, \tag{3.21}$$

$$b' + c\left(1 + W^2\cos(\Delta\varphi)\,\exp\left(-\frac{c^2W^2}{2} - \frac{2b^2}{W^2}\right)\right) = 0, \tag{3.22}$$

$$\sin(\Delta\varphi)\,\exp\left(-\frac{c^2W^2}{2} - \frac{2b^2}{W^2}\right) = 0, \tag{3.23}$$

$$\frac{NW}{\sqrt{\pi}} - 4 + 4\cos(\Delta\varphi)\,\left(c^2W^4 - 4b^2\right)\exp\left(-\frac{c^2W^2}{2} - \frac{2b^2}{W^2}\right) = 0. \tag{3.24}$$

The equation for the total phase φ is just a conservation of the number of atoms in the condensate. If the magnitudes of $b(t)$ and $c(t)$ are very small, we can adopt the approximation that $\exp\left(-\frac{c^2W^2}{2} - \frac{2b^2}{W^2}\right) \approx 1$. Consequently, the linearized forms of the above Euler-Lagrange equations simplify dramatically

$$b' \approx -c(1+W^2), \tag{3.25}$$

$$c' \approx \frac{4b}{W^2}, \tag{3.26}$$

where the equations of width and relative phase lead to $W \approx W_0 = \frac{4\sqrt{\pi}}{N}$ and $\Delta\varphi \approx \Delta\varphi_0 = 0$, which are actually the values at stationary states. Eliminating the parameter $c(t)$, we obtain the equation for $b(t)$ as follows

$$b'' + \frac{4(1+W_0^2)}{W_0^2} b = 0, \tag{3.27}$$

or

$$b'' + \Omega_2^2 b = 0. \tag{3.28}$$

Hence, parameter $b(t)$ oscillates with frequency

$$\Omega_2 = 2\sqrt{1+\frac{N^2}{16\pi}}. \tag{3.29}$$

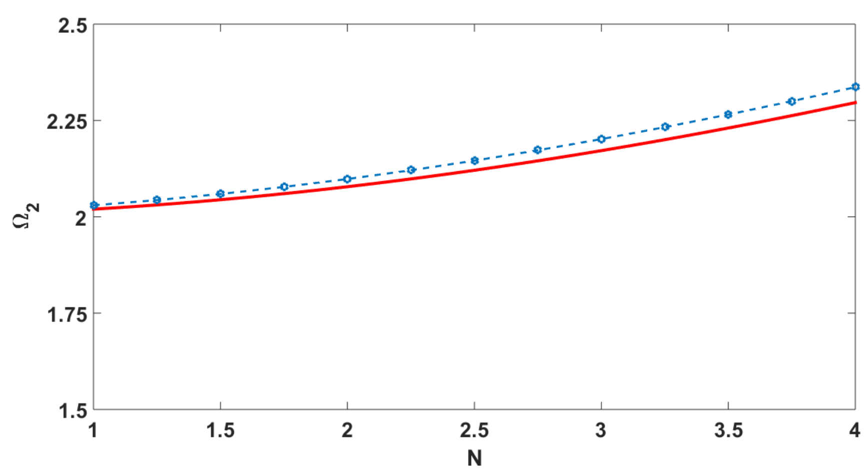

Figure 6. The frequency of the second collective mode of Bose-Einstein condensate as a function of norm N. The red curve is the prediction of variation approximation (3.29) while the dots are results from diagonalization of Bogoliubov-de Gennes equation (2.15)

In Figure 6, we present a comparison between the predictions of the variational approximation and the numerical results obtained using the Bogoliubov method. It can be seen that the two results are in good agreement.

The dynamics of the BEC under excitation of this mode, within the variational approximation using the ansatz (3.18)–(3.19), shows oscillations of the condensate components about their centers. This behavior is confirmed by direct simulations (see Fig. 4), where the two components exhibit out-of-phase oscillations along the x-axis during the evolution. Since no external trapping is applied along the direction of oscillation, this mode differs from the well-known dipole oscillation, which obeys the Kohn theorem [13]. This special collective excitation arises from the interplay between the inter-channel coupling and the attractive interactions within the condensate.

4. Conclusions

In this study, we analyze in detail the collective mode structure of a dilute BoseEinstein condensate trapped in a two-channel potential. We employ the variational approximation and the Bogoliubov method to determine the mode frequencies and the corresponding shape functions. The results obtained from the two approaches are in good agreement. In addition, direct numerical simulations of the real-time quantum dynamics of the condensate were performed to confirm several conclusions drawn from the approximate methods.

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REFERENCES

[1] Bose SN, (1924). *Zeitschrift fr Physik - Journal for Physics* 26, 178; Einstein A, (1925). *Sitzber. Kgl. Preuss. Akad. Wiss.* 23, 3. DOI: 10.1007/BF01327326

[2] Ketterle W, (2002). Bose-Einstein condensation and the atom laser. *Reviews of Modern Physics*, 74, 1131. DOI: 10.1103/RevModPhys.74.1131

[3] Cornell EA and Wieman CE, (2002). Bose-Einstein condensation in a dilute gas, the first 70 years and some recent experiments. *Reviews of Modern Physics*, 74, 875. DOI: 10.1103/RevModPhys.74.875

[4] Pethick CJ and Smith H, (2010). *Bose-Einstein Condensation of dilute gases*. Cambridge University Press, England.

[5] Ueda M, (2010). *Fundamentals And New Frontiers Of Bose-Einstein Condensation*. World Scientific Publishing.

[6] Dalfovo F, Giorgini S, Pitaevskii LP and Stringari S, (1999). Theory of Bose-Einstein condensation in trapped gases. *Reviews of Modern Physics*, 71, 463. DOI: 10.1103/RevModPhys.71.463

[7] Leggett AJ, (2001). Bose-Einstein condensation in the alkali gases: Some fundamental concepts. *Reviews of Modern Physics*, 73, 307. DOI: 10.1103/RevModPhys.73.307

[8] Nguyen VH, Matuszevski M, Trippenbach M, (2009). Matter wave soliton collisions in the quasi one-dimensional potential. *Physica D*, 238, 1449. DOI: 10.1016/j.physd.2008.07.023

[9] Nguyen VH, Zi P, Infeld E, Trippenbach M, (2014). Symmetry breaking in the collisions of double channel BEC solitons. *Physica D*, 269, 37. DOI: 10.1016/j.physd.2013.11.011

[10] Nguyen VH, Sanzkowski P, Konotop VV and Trippenbach M, (2020). Four-wave mixing in spin-orbit coupled Bose-Einstein Condensates. *New Journal of Physics*, 2, 5, 053019. DOI: 10.1088/1367-2630/ab7fcb

[11] Hui D, Hartmut H, and Yoshihisa Y, (2010). Exciton-polariton Bose-Einstein condensation. *Reviews of Modern Physics*, 82, 1489. DOI: 10.1103/RevModPhys.82.1489

[12] Zegadlo K, Nguyen VH, Konotop VV, Zakrzewski J, and Trippenbach M, (2019). Route to chaos in a coupled micro-resonator system with gain and loss. *Nonlinear Dynamics* 97, 559. DOI: 10.1007/s11071-019-04997-5.

[13] Jin DS, Ensher JR, Matthews MR, Wieman CE, and Cornell EA, (1996). Collective Excitations of a Bose-Einstein Condensate in a Dilute Gas. *Physical Review Letters*, 77, 420. DOI: 10.1103/PhysRevLett.77.420.

[14] Zhenlian S, Ziliang L, Pengjun W, Wei H, Lianghui H, Zengming M, Liangchao C and Jing Z, (2023). Collective excitation of BoseEinstein condensate of 23Na via high-partial wave Feshbach resonance. *New Journal of Physics*, 25, 023032. DOI: 10.1088/1367-2630/acbd67.

[15] Saswata S and Dwipesh M, (2020). Collective excitations of two-dimensional Bose-Einstein Condensate in liquid phase with spin-orbit coupling. *Journal of Physics B: Atomic, Molecular and Optical Physics,* 53, 095301. DOI: 10.1088/1361-6455/ab7528.

[16] Pedro CD, Eduardo ABO, Aristeu RPL and Emanuel ALH, (2020). Ground state and collective excitations of a dipolar Bose-Einstein condensate in a bubble trap. *Scientific Reports* 10, 4831. DOI: 10.1038/s41598-020-61657-0

[17] Wachtler F and Santos L, (2016). Ground-state properties and elementary excitations of quantum droplets in dipolar Bose-Einstein condensates. *Physical Review A* 94, 043618. DOI: 10.1103/PhysRevA.94.043618

[18] ZhaoHui W and JiXuan H, (2020). LowLying Collective Modes of a OneDimensional Bose Gas with Quantum Fluctuation Effect. *Journal of Low Temperature Physics,* 199, 13241331, DOI: 10.1007/s10909-020-02438-x

[19] Anna G, Maria ET, Vincenzo A, Fabrizio R, Milena DG, Dimitris T, Kirk B, Loren P, Dario B, Daniele S and Nina V, (2023). Collective excitations of a bound-in-the continuum condensate. *Nature Communications* 14, 3464. DOI: 10.1038/s41467-023-38939-y

[20] Huang GX, (2004). Second harmonic generation of propagating collective

excitations in BoseEinstein condensates. *Chinese Physics,* 13, 1866. DOI: 1009-1963/13/11/018

[21] Momme MR, Prikhodko OO, and Bidasyuk YM, (2020). Dispersion relations and self-localization of quasiparticles in coupled elongated Bose-Einstein condensates. *Physical Review A*, 102, 043316. DOI: 10.1103/PhysRevA.102.043316

[22] Otajonov SR, Abdullaev FK, Shermaxmatov A, (2026). Beyond mean-field effects in Josephson oscillations and self-trapping of BoseEinstein condensates in two-dimensional dual-core traps. *Chaos, Solitons and Fractals* 202, 117481. DOI: 10.1016/j.chaos.2025.117481

[23] Nguyen VH, (2010). *Bight solitons in one- and two-dimensional Kerr media. External potential versus nonlinearity modulation*. Ph.D. thesis at Soltan Institute for Nuclear Studies, Poland.

[24] Bogoliubov NN, (1947). On the theory of superfluidity. *Journal of Physics (USSR)* 11, 23. DOI: 10.1016/b978-0-08-015816-7.50020-1

[25] Moin P, (2010). *Fundamentals of engineering numerical analysis*, Cambridge University Press.

[26] Malomed BA, (2002). Variational methods in nonlinear fiber optics and related fields. *Progress in Optics* 43, 71 (ed. by E. Wolf: North Holland, Amsterdam). DOI: 10.1016/S0079-6638(02)80026-9

[27] Ramaniuk A, Hung NV, Giersig M, Kempa K, Konotop VV, Trippenbach M, (2018). Vortex creation without stirring in coupled ring resonators with gain and loss. *Symmetry* 10, 195 . DOI: 10.3390/sym10060195

[28] Nguyen DC, Dinh XK, Le XTT, Nguyen VH, Trippenbach M, (2019). On the nonlinear dynamics of coupled micro-resonators. *Proceedings of SPIE* 11204, 68-73 (2019). DOI: 10.1117/12.2537262

[29] Nguyen VH, Zegadlo K, Ramaniuk A, Konotop VV, Trippenbach M, (2017). Modulational instability of coupled ring waveguides with linear gain and nonlinear loss. *Scientific Reports*, 7, 4089. DOI: 10.1038/s41598-017-04408-y

[30] Nguyen DQH, Tran PY, Do DT, Cao XT, Bui DT, Tran KV, Mattia L, Nguyen TD, Tran TH, Nguyen VH, (2024). All-optical logic gate operation with a multi-core nonlinear photonic crystal fiber. *HNUE Journal of Science: Natural Science* 69, (2), pp. 47-57. DOI: 10.18173/2354-1059.2024-0019.