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PROPER HOLOMORPHIC MAPS AND $B-$REGULAR DOMAINS IN $\mathbb{C}^n$

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Abstract. We begin by proving that B-regularity is preserved under proper holomorphic mappings of domains. We next investigate B-regularity in the setting of bounded Reinhardt domains in $\mathbb{C}^n$. These results together demonstrate that B-regularity can be carried from well-understood Reinhardt model domains to more complicated domains via explicit proper holomorphic maps.
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1. Introduction

Let Ω be a bounded domain in $\mathbb{R}^n$. Denote by $C(\partial\Omega)$ and $C(\Omega)$ the spaces of real-valued continuous functions on $\partial\Omega$ and Ω, respectively. A classical problem in (real) potential theory asks: *Given $f \in C(\partial\Omega)$, does there exist $u \in C(\Omega)$ harmonic on Ω with $u|_{\partial\Omega} = f$?*

It is well known that, whenever such an extension exists, it can be obtained as the Perron envelope

$$u_{f,\Omega} := \sup\big\{v \in \mathrm{SH}(\Omega) : v^* \leq f \text{ on } \partial\Omega\big\},$$

where v^* is the upper semicontinuous regularization. The function $u_{f,\Omega}$ is harmonic in Ω even if f is merely bounded. Moreover, the Perron method solves the Dirichlet problem *if and only if* each boundary point admits a (subharmonic) barrier.

It is natural to ask for a complex analogue. If $\Omega \subset \mathbb{C}^n$ is bounded, we want conditions ensuring that every $f \in C(\partial\Omega)$ extends to a *maximal* plurisubharmonic function u on Ω which is continuous up to $\partial\Omega$. Recall: $u \in \mathrm{PSH}(\Omega)$ is maximal if for every $\Omega' \Subset \Omega$ and every $v \in \mathrm{PSH}(\Omega')$ with $v^* \leq u$ on $\partial\Omega'$ we have $v \leq u$ in Ω'.

Bremermann [1] showed that a natural candidate is the Perron–Bremermann envelope

$$u_{f,\Omega} := \sup\big\{v \in \mathrm{PSH}(\Omega) : v^* \leq f \text{ on } \partial\Omega\big\}.$$

Unlike the real case, $u_{f,\Omega}$ may fail to be upper semicontinuous. Walsh [2] proved that to ensure continuity it suffices to verify the boundary limits

$$\lim_{z \to x} u_{f,\Omega}(z) = f(x) \qquad \forall x \in \partial\Omega.$$

Sibony [3] (see also [4], [5]) developed powerful criteria for when the Perron–Bremermann method works uniformly. Moreover, bounded domains in $\mathbb{C}^n$ on which Dirichlet problem for maximal plurisubharmonic functions is termed B-regular, according to [3]. Recent developments have expanded the scope of the theory of B-regular domains beyond the classical Dirichlet problem for continuous boundary data. Nilsson and Wikström [6] introduced the class of quasibounded plurisubharmonic functions and applied Jensen measure techniques to generalized Dirichlet problems with unbounded boundary values. Nilsson [7] further studied Perron–Bremermann envelopes on bounded B-regular domains and established continuity results for envelopes associated with certain unbounded boundary data. More recently, Nilsson [8] investigated plurisubharmonic functions with discontinuous boundary behavior and proved the existence and uniqueness results for the complex Monge–Ampère Dirichlet problem with bounded discontinuous boundary values on B-regular domains. These works show that the modern study of B-regularity is closely connected with refined boundary value problems, envelope constructions, and singular boundary phenomena in pluripotential theory.

In this note, we emphasize that B-regularity behaves well under *proper holomorphic mappings*. Recall that a holomorphic map $\pi : D \to \Omega$ between bounded domains is *proper* if $\pi^{-1}(K)$ is compact in D for every compact $K \subset \Omega$. Proper maps appear naturally as finite branched coverings and are ubiquitous in symmetry reductions and model constructions. Our first main result (Theorem 2.2) shows that, under a mild boundary extension assumption, B-regularity is an *invariant* of proper holomorphic equivalence: if $\pi : D \to \Omega$ is proper holomorphic and extends continuously to the closures, then

$$D \text{ is } B\text{-regular} \quad \Longleftrightarrow \quad \Omega \text{ is } B\text{-regular}.$$

This invariance principle is a convenient tool for producing new B-regular examples by pulling back from domains with well-understood geometry, and it is also a conceptual explanation for why certain "branched models" inherit boundary regularity.

The second focus of the paper is the class of *Reinhardt domains*. A bounded domain $\Omega \subset \mathbb{C}^n$ is Reinhardt if it is invariant under independent rotations in each coordinate. Such domains reduce many analytic questions to convex geometry through the logarithmic image $\log \Omega^* \subset \mathbb{R}^n$, and they form a natural testing ground for fine boundary regularity phenomena. Our main structural statement here is Proposition 3.1: for a bounded Reinhardt domain Ω, B-regularity is equivalent to the conjunction of

- hyperconvexity (existence of a negative plurisubharmonic exhaustion), and
- absence of analytic structure in the boundary, i.e. $\partial\Omega$ contains no nontrivial analytic disks.

The proof exploits the torus symmetry to normalize boundary points, then uses the geometry of the logarithmic image to build local barriers; if a barrier construction fails, one is forced to produce a nonconstant analytic disk in $\partial\Omega$, contradicting the hypothesis.

Finally, combining Proposition 3.1 with Theorem 2.2 yields a flexible mechanism: first verify B-regularity for a Reinhardt model domain (often by a short "no analytic disk" argument), then transport the property to more complicated domains via an explicit proper holomorphic map.

2. Preliminaries

We start with the following basic notion about domains admitting bounded exhaustion plurisubharmonic functions. This type of domains play central role in solving Dirichlet problem with respect to the complex Monge-Ampère equation.

Definition 2.1. *A bounded open set $\Omega \subset \mathbb{C}^n$ is called* hyperconvex *if there exists a negative plurisubharmonic exhaustion function for Ω.*

Kerzman–Rosay [9] showed that every bounded pseudoconvex domain with C^1 boundary is hyperconvex, and every hyperconvex domain admits a negative C^∞ strictly plurisubharmonic exhaustion function (see also [4]). A substantial class of hyperconvex domains is described in the definition below.

Definition 2.2. *A bounded domain $\Omega \subset \mathbb{C}^n$ is B-*regular *if every real-valued $f \in C(\partial\Omega)$ admits an extension $u \in \mathrm{PSH}(\Omega) \cap C(\overline{\Omega})$.*

The following theorem yields a very useful practical criterion to realize $B-$regularity.

Theorem 2.1. *For a bounded domain $\Omega \subset \mathbb{C}^n$, the following are equivalent:*

(i) Ω *is B-regular.*

(ii) *For every $p \in \partial\Omega$ there exists a strong barrier at p w.r.t. Ω: there is $u \in \mathrm{PSH}(\Omega) \cap C(\overline{\Omega})$ such that $u(p) = 0$ and $u < 0$ on $\overline{\Omega} \setminus \{p\}$.*

(iii) *For every $p \in \partial\Omega$ there exists a weak barrier at p: there exist $\varphi \in \mathrm{PSH}(\Omega), \varphi < 0$ such that $\lim\limits_{z\to p} \varphi(z) = 0$ and $\limsup\limits_{z\to q} \varphi(z) < 0$ for $q \in (\partial\Omega) \setminus \{p\}$.*

(iv) For every $p \in \partial\Omega$ there exists a local barrier at p: there exist an open neighborhood U of p and a plurisubharmonic function u on $\Omega \cap U$ and continuous on $\overline{\Omega \cap U}$ such that $u < 0$ on $\overline{\Omega \cap U} \setminus \{p\}$ and $u(p) = 0$.

The main result in our note is the following invariant property of B-regularity under proper holomorphic maps

Theorem 2.2. *Let D, Ω be bounded domains in $\mathbb{C}^n$. Assume that $\pi : D \to \Omega$ is a proper holomorphic mapping which extends to a continuous mapping between the closures of the two domains. Then D is B-regular if and only if Ω is B-regular.*

Proof. We use the barrier characterization given in Theorem 2.1: a bounded domain is B-regular if and only if every boundary point admits a plurisubharmonic barrier.
(⇐) Assume Ω is B-regular. Fix $z_0 \in \partial D$ and set $w_0 = \pi(z_0) \in \partial\Omega$. By Theorem 2.1, there exists a function $\psi \in \mathrm{PSH}(\Omega) \cap C(\overline{\Omega})$ such that $\psi(w_0) = 0$ and $\psi < 0$ on $\overline{\Omega} \setminus \{w_0\}$. Then $\varphi := \psi \circ \pi$ belongs to $\mathrm{PSH}(D)$ (since π is holomorphic) and is continuous on $\overline{D}$ (since π extends continuously to $\overline{D}$). Moreover, $\varphi(z_0) = \psi(w_0) = 0$ and $\varphi < 0$ on $\overline{D} \setminus \{z_0\}$. Thus, φ is a barrier at z_0. Since z_0 is arbitrary, D is B-regular by Theorem 2.1.
(⇒) Assume D is B-regular. Fix $w_0 \in \partial\Omega$. Choose $z_0 \in \partial D$ with $\pi(z_0) = w_0$ (which exists because $\pi(\overline{D}) = \overline{\Omega}$). By Theorem2.1 there exists an $u \in \mathrm{PSH}(D) \cap C(\overline{D})$ such that $u(z_0) = 0, u < 0$ on $\bar{D} \setminus \{z_0\}$.

We now define on Ω the fiberwise maximum

$$v(w) := \max\{u(z) : z \in \pi^{-1}(w)\}, \qquad w \in \Omega. \tag{2.1}$$

Since π is holomorphic and proper between bounded domains, for each $w \in \Omega$, the fiber $\pi^{-1}(w)$ consists of finitely many points counted with multiplicities. So there is $E \subset \Omega$, an analytic set of critical values (possibly empty) of π such that on $\Omega \setminus E$ the map π is a finite unbranched covering of degree m. Hence locally on $\Omega \setminus E$, one can write π^{-1} as m holomorphic inverse branches. Thus locally around any point in $\Omega \setminus E$, the function v is maximum of m plurisubharmonic functions, and therefore, $v \in \mathrm{PSH}(\Omega \setminus E)$.

Moreover, $\widetilde{u} \leq 0$ on $\overline{D}$, so $v \leq 0$ on Ω, hence v is locally bounded above near E. By removability of plurisubharmonic singularities across analytic sets, v extends uniquely to a function in $\mathrm{PSH}(\Omega)$, also denoted by v. We now claim that v peaks at w_0. Let $w_\nu \to w_0$ with $w_\nu \in \Omega$. Since $\overline{D}$ is compact and π is continuous on $\overline{D}$, after passing to a subsequence, we can choose $z_\nu \in \pi^{-1}(w_\nu)$ such that $z_\nu \to z_0$ (this uses that $z_0 \in \pi^{-1}(w_0)$ and fibers are finite in the interior; properness prevents mass from escaping). Then, $u(z_\nu) \to u(z_0) = 0$. From (2.1) we obtain $v(w_\nu) \geq \widetilde{u}(z_\nu)$. This yields $\liminf_{\nu\to\infty} v(w_\nu) \geq 0$. On the other hand $v \leq 0$ everywhere, hence along this subsequence $v(w_\nu) \to 0$. Since $w_0 \in \partial\Omega$ is arbitrary, every boundary point of Ω admits a barrier, hence Ω is B-regular by Theorem 2.1. □

The assumption of continuity up to the boundary for the maps is crucial, as it allows one to transfer peak plurisubharmonic functions from one domain to the other. We do not currently see how the argument could proceed in the absence of this assumption.

3. Results

We now treat bounded Reinhardt domains and give a transparent proof of a criterion for $B-$regularity.

Proposition 3.1. *Let $\Omega \subset \mathbb{C}^n$ be a bounded Reinhardt domain. Then, Ω is B-regular if and only if Ω is hyperconvex and $\partial\Omega$ contains no analytic structure, i.e., every holomorphic mapping from the unit disk in to $\partial\Omega$ is constant.*

Recall that Ω is Reinhardt if $(e^{i\theta_1}z_1, \ldots, e^{i\theta_n}z_n) \in \Omega$ whenever $(z_1, \ldots, z_n) \in \Omega$ and $(\theta_1, \ldots, \theta_n) \in \mathbb{R}^n$. Also set

$$V_j = \{z \in \mathbb{C}^n : \ z_j = 0\}, \qquad V = \bigcup_{j=1}^{n} V_j,$$

and

$$\log \Omega^* = \{(\log |z_1|, \ldots, \log |z_n|) : z \in \Omega \setminus V\} \subset \mathbb{R}^n.$$

This result was established in Proposition 3.1 of [10]. The original proof of Proposition 3.1 in [10] is rather brief and somewhat difficult to follow. The main contribution of our proof is therefore to elaborate on the arguments in [10], in particular, the construction of plurisubharmonic barriers at boundary points. To proceed, we recall the following known criteria for verifying pseudoconvexity and hyperconvexity of Reinhardt domains (see [11]):

Lemma 3.1. *Let Ω be a bounded Reinhardt. Then Ω is pseudoconvex if and only if:*

(a) $\log \Omega^*$ *is convex in $\mathbb{R}^n$, and*

(b) whenever $\Omega \cap V_j \neq \varnothing$, the domain is radially monotone in the j-th coordinate: $(z_1, \ldots, z_j, \ldots, z_n) \in \Omega \Rightarrow (z_1, \ldots, \lambda z_j, \ldots, z_n) \in \Omega$ *for all* $|\lambda| < 1$.

Lemma 3.2. *Let Ω be a bounded pseudoconvex Reinhardt domain. Then, Ω is hyperconvex if and only if $\bar{\Omega} \cap V_j \neq \emptyset \Rightarrow \Omega \cap V_j \neq \emptyset$.*

Proof of Proposition 3.1. We prove the nontrivial implication "$\Leftarrow$". Assume that $\Omega \subset \mathbb{C}^n$ is a bounded Reinhardt domain, Ω is hyperconvex, and $\partial\Omega$ contains no analytic structure. By Theorem 2.1, it is sufficient to show that *every* boundary point admits a *local barrier*.

Fix an arbitrary point $a \in \partial\Omega$. Since Ω is Reinhardt, we may rotate each coordinate independently. In particular, after multiplying the j-th coordinate by a unimodular constant, we may assume that each nonzero component of a is a positive real number. Next, by a diagonal complex linear change of variables (rescaling coordinates), we may normalize these positive real numbers to 1. Therefore, without loss of generality, we may assume

$$a = (\underbrace{1, \dots, 1}_{k \text{ times}}, \underbrace{0, \dots, 0}_{n-k \text{ times}}) \qquad \text{for some } 0 \le k \le n.$$

Because Ω is hyperconvex, the "degenerate" situation $k = 0$ is excluded for bounded pseudoconvex Reinhardt domains (by Zwonek's hyperconvexity criterion, Lemma 3.2); hence we may and do assume $1 \le k \le n$. We consider two cases separately:

Case 1 $k = n$. In this case $a \notin V = \bigcup_{j=1}^n \{z_j = 0\}$, so we can work in a small neighborhood where all coordinates stay nonzero. Since Ω is hyperconvex it is in particular pseudoconvex, and by Lemma 3.1 the set

$$\log \Omega^* = \{(\log |z_1|, \dots, \log |z_n|) : z \in \Omega \setminus V\} \subset \mathbb{R}^n$$

is convex. Because $|a_j| = 1$ for all j, we have $(0, \dots, 0) \in \partial(\log \Omega^*)$.

By the supporting hyperplane theorem for convex sets, there exists a nonzero vector $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{R}^n$ such that the affine functional $x \mapsto \langle \alpha, x \rangle$ supports $\log \Omega^*$ at the origin, i.e.

$$\langle \alpha, x \rangle \le 0 \quad \text{for all } x \in \log \Omega^*, \qquad \text{and} \qquad \langle \alpha, 0 \rangle = 0.$$

Choose a neighbourhood U of a so small that $U \cap V = \varnothing$. Define on $\Omega \cap U$

$$\phi(z) := \sum_{j=1}^n \alpha_j \log |z_j|.$$

This function is plurisubharmonic on $\Omega \cap U$ (each $\log |z_j|$ is psh on $\{z_j \neq 0\}$, and sums preserve psh), and it is continuous on $\Omega \cap U$ because no coordinate vanishes there. Moreover, for $z \in \Omega \cap U$ we have $(\log |z_1|, \dots, \log |z_n|) \in \log \Omega^*$, so

$$\phi(z) = \big\langle \alpha, (\log |z_1|, \dots, \log |z_n|) \big\rangle \le 0, \qquad \text{and} \qquad \phi(a) = \sum_{j=1}^n \alpha_j \log 1 = 0.$$

To see that ϕ is a *local barrier* at a, we must check the strict inequality $\phi < 0$ on $(\overline{\Omega} \cap U) \setminus \{a\}$; in particular on $(\partial\Omega \cap U) \setminus \{a\}$.

Suppose, for a contradiction, that there exists a $b \in (\partial\Omega \cap U) \setminus \{a\}$ with $\phi(b) = 0$. Since $U \cap V = \varnothing$, we have $b \notin V$, hence $w := (\log |b_1|, \dots, \log |b_n|) \in \partial(\log \Omega^*)$. The equality $\phi(b) = 0$ means exactly that $\langle \alpha, w \rangle = 0$, i.e. w lies in the supporting hyperplane.

Now consider the holomorphic map

$$\Psi : \mathbb{C} \to \mathbb{C}^n, \qquad \Psi(\zeta) = \big(e^{w_1 \zeta}, \dots, e^{w_n \zeta}\big).$$

For any ζ we compute

$$\log|\Psi(\zeta)| = (\log|e^{w_1\zeta}|, \ldots, \log|e^{w_n\zeta}|) = \big((\mathrm{Re}\zeta)w_1, \ldots, (\mathrm{Re}\zeta)w_n\big) = (\mathrm{Re}\zeta)\, w.$$

Hence for ζ in the vertical strip $H = \{\zeta \in \mathbb{C} : 0 < \mathrm{Re}\zeta < 1\}$ we get $(\mathrm{Re}\zeta)\, w$ with $0 < \mathrm{Re}\zeta < 1$.

Because $\log\Omega^*$ is convex and α is a supporting functional with $\langle\alpha, 0\rangle = \langle\alpha, w\rangle = 0$, the whole segment $\{tw : 0 \le t \le 1\}$ lies in the *supporting hyperplane* $\{\langle\alpha, x\rangle = 0\}$. Moreover, since $\log\Omega^*$ lies in $\{\langle\alpha, x\rangle \le 0\}$ and contains points arbitrarily close to 0, this segment is contained in $\partial(\log\Omega^*)$. Consequently,

$$\log|\Psi(\zeta)| = (\mathrm{Re}\zeta)\, w \in \partial(\log\Omega^*) \quad \Longrightarrow \quad \Psi(\zeta) \in \partial\Omega \qquad (\zeta \in H).$$

Therefore, $\Psi(H) \subset \partial\Omega$. Since $w \neq 0$ (because $b \neq a$), the map Ψ is non-constant, and thus we have exhibited a non-constant holomorphic image contained in $\partial\Omega$, i.e. an analytic structure in the boundary. This contradicts with the assumption on $\partial\Omega$. Hence such a b cannot exist, and we conclude $\phi < 0$ on $(\partial\Omega \cap U) \setminus \{a\}$. Thus ϕ is a local barrier at a.

Case 2 $1 \le k < n$ i.e., some coordinates vanish at a. Let $\pi : \mathbb{C}^n \to \mathbb{C}^k$ be the coordinate projection $\pi(z_1, \ldots, z_n) = (z_1, \ldots, z_k)$. Using that Ω is Reinhardt and pseudoconvex (hyperconvex $\Rightarrow$ pseudoconvex), one checks from Lemma 3.1 that $\pi(\Omega)$ is also a bounded pseudoconvex Reinhardt domain in $\mathbb{C}^k$. Moreover, since Ω is hyperconvex, Zwonek's criterion (Lemma 3.2) ensures that $\pi(\Omega)$ is hyperconvex as well (in particular, the projection does not introduce a degeneracy along coordinate hyperplanes).

We claim that $\partial(\pi(\Omega))$ has no analytic structure. Indeed, if there were a non-constant holomorphic disc $F : \Delta \to \mathbb{C}^k$ with $F(\Delta) \subset \partial(\pi(\Omega))$, then the map

$$\widetilde{F} : \Delta \to \mathbb{C}^n, \qquad \widetilde{F}(\zeta) = (F(\zeta), 0, \ldots, 0)$$

would be non-constant and would satisfy $\widetilde{F}(\Delta) \subset \partial\Omega$, producing analytic structure in $\partial\Omega$. This raises clearly a contradiction.

Now note that $\pi(a) = (1, \ldots, 1) \in \partial(\pi(\Omega))$. Applying the argument of Step 2 in dimension k to the hyperconvex Reinhardt domain $\pi(\Omega)$ (whose boundary has no analytic structure), we obtain a neighbourhood U_0 of $\pi(a)$ in $\mathbb{C}^k$ and a local barrier $u \in \mathrm{PSH}(\pi(\Omega) \cap U_0) \cap C(\overline{\pi(\Omega) \cap U_0})$ such that $u(\pi(a)) = 0$ and $u < 0$ on $(\overline{\pi(\Omega)} \cap U_0) \setminus \{\pi(a)\}$. Then, $u \circ \pi$ is plurisubharmonic and continuous on $\Omega \cap \pi^{-1}(U_0)$, and it satisfies

$$(u \circ \pi)(a) = 0, \qquad u \circ \pi < 0 \text{ on } (\overline{\Omega} \cap \pi^{-1}(U_0)) \setminus \{a\},$$

so $u \circ \pi$ is a local barrier at a for Ω.

Since $a \in \partial\Omega$ is arbitrary, every boundary point admits a local barrier. By Theorem 2.1, Ω is B-regular. □

Corollary 3.1. *Let $a_1, \dots, a_n$ be positive constants. The domain*

$$\Omega = \left\{ z \in \mathbb{C}^n : \ |z_1|^{a_1} + \cdots + |z_n|^{a_n} < 1 \right\}$$

is B-regular.

Proof. First, Ω is clearly bounded and Reinhardt, since the defining function depends only on the moduli $|z_j|$. To prove hyperconvexity, consider the function

$$\rho(z) := \sum_{j=1}^{n} |z_j|^{a_j} - 1.$$

Then $\rho < 0$ on Ω and $\rho = 0$ on $\partial\Omega$. Moreover, since $z \mapsto |z_j|^{a_j}$ is plurisubharmonic on $\mathbb{C}^n$ (it depends on one complex variable only and is subharmonic there), we infer that the sum ρ is plurisubharmonic on $\mathbb{C}^n$ and in particular on Ω. Thus, ρ is a negative plurisubharmonic exhaustion on Ω.

Next we show that $\partial\Omega$ has no analytic structure. Assume for a contradiction that there exists a non-constant holomorphic map

$$\Phi = (\phi_1, \dots, \phi_n) : \Delta \to \mathbb{C}^n \qquad \text{with} \qquad \Phi(\Delta) \subset \partial\Omega.$$

Then, for every $\zeta \in \Delta$,

$$\sum_{j=1}^{n} |\phi_j(\zeta)|^{a_j} = 1. \tag{3.1}$$

Since Φ is non-constant, at least one component is non-constant. Let $J := \{j : \phi_j \not\equiv 0\}$. Pick a point $\zeta_0 \in \Delta$ where $\phi_j(\zeta_0) \neq 0$ for all $j \in J$ (such a point exists because zeros of a nonzero holomorphic function are discrete). Shrinking to a small disc $\Delta' \Subset \Delta$ around ζ_0, we may assume that each ϕ_j ($j \in J$) is nowhere zero on Δ'. Then on Δ' we can write

$$\phi_j = e^{\psi_j} \qquad (j \in J),$$

with ψ_j holomorphic on Δ'. Substituting into (3.1) gives, on Δ',

$$\sum_{j \in J} e^{a_j \mathrm{Re}\psi_j} = 1,$$

because $|e^{\psi_j}| = e^{\mathrm{Re}\psi_j}$.

Now apply $\frac{\partial^2}{\partial\zeta\,\partial\overline{\zeta}}$ to both sides. Since $\mathrm{Re}\psi_j$ is harmonic, using the chain rule, we get

$$\frac{\partial^2}{\partial\zeta\,\partial\overline{\zeta}} e^{a_j \mathrm{Re}\psi_j} = a_j^2 \left| \frac{\partial}{\partial\zeta} \mathrm{Re}\psi_j \right|^2 e^{a_j \mathrm{Re}\psi_j}.$$

Moreover, because ψ_j is holomorphic,

$$\frac{\partial}{\partial\zeta}\mathrm{Re}\psi_j = \frac{1}{2}\,\psi_j'(\zeta),$$

hence

$$\frac{\partial^2}{\partial\zeta\,\partial\overline{\zeta}}\,e^{a_j\mathrm{Re}\psi_j} = \frac{a_j^2}{4}\,|\psi_j'(\zeta)|^2\,e^{a_j\mathrm{Re}\psi_j} \;\geq 0.$$

Therefore, differentiating the identity yields, for all $\zeta \in \Delta'$,

$$0 = \frac{\partial^2}{\partial\zeta\,\partial\overline{\zeta}}\Big(\sum_{j\in J} e^{a_j\mathrm{Re}\psi_j}\Big) = \sum_{j\in J}\frac{a_j^2}{4}\,|\psi_j'(\zeta)|^2\,e^{a_j\mathrm{Re}\psi_j}.$$

Each summand is nonnegative, so every term must vanish identically on Δ'. It follows that Φ is a constant on Δ, contradicting with the assumption that Φ is non-constant. Hence, $\partial\Omega$ contains no analytic structure. We have proved that Ω is a bounded hyperconvex Reinhardt domain whose boundary has no analytic structure. By Proposition 3.1, Ω is B-regular. □

Corollary 3.2. *The domain*

$$\Omega := \{(z_1, z_2) \in \mathbb{C}^2 : |z_1|^2 + |z_2^2 + z_2| < 1\}$$

is bounded $B-$ *regular in* $\mathbb{C}^2$.

Proof. Let $\pi : \mathbb{C}^2 \to \mathbb{C}^2$ be the holomorphic mapping

$$\pi(z_1, z_2) = (z_1^2, z_2^2 + z_2).$$

We show that π is proper. To see this, it suffices to note that $|z_2^2 + z_2| > |z_2|^2/2$ for $|z_2| > 2$. Thus $\pi : \Omega \to D$ is a proper holomorphic mapping. Finally, by Corollary 3.1, the Reinhardt domain $D := \{(z_1, z_2) : |z_1| + |z_2| < 1\}$ is B-regular, so using Theorem 2.2 we conclude that Ω is $B-$regular as well. □

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