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# ON CONVEX COMBINATIONS OF GRAZING LINEAR MODES FOR 2-DEGREE-OF-FREEDOM VIBRO-IMPACT SYSTEMS WITH 1:2 RESONANCE

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**Abstract.** Grazing periodic orbits in vibro-impact mechanical systems are known to generate nonlinear normal modes. In the absence of internal resonance, many of the essential dynamical features already appear in two-degree-of-freedom (2-dof) mass–spring chains. In this note, we focus on such a system under the simplest internal resonance condition. Two grazing linear modes, denoted as $\mathrm{GLOM}_1$ and $\mathrm{GLOM}_2$, associated with the first and the second linear modes, play a key role.We will show that the grazing linear orbit generating one-impact-per-period (1IPP) solutions is neither $\mathrm{GLOM}_1$ nor $\mathrm{GLOM}_2$, but rather a convex combination of the two. This leads to the introduction of the set $\mathrm{GLO}_c = [\mathrm{GLOM}_1, \mathrm{GLOM}_2]$ of grazing linear orbits. For all $\mathrm{GLO}_c$ except the endpoint modes, the free-flight time to impact admits a particularly simple structure, providing a convenient framework for the future study of the nonlinear behavior of such grazing linear orbits.
***Keywords:*** unilateral contact, Grazing linear mode, free flight time, internal resonances.

## 1. Introduction

Grazing orbits are known to introduce significant dynamical effects in vibro-impact systems [1]-[3]. They may generate square-root singularities in the return map [4], induce grazing bifurcations [2], and trigger chaotic dynamics [5], [6]. These phenomena arise from the intrinsic non-smoothness of unilateral contact and fundamentally shape the behavior of impact oscillators.

Although grazing linear orbits are rare [7], [8], they organize the main branches of nonlinear normal modes [9], [10]. Many phenomena observed in $N$-dof systems, particularly the structure of the nonsmooth modal spectrum [11], [12], already appear in the 2-dof case. Thus, 2-dof mass–spring chains provide a minimal framework for capturing the essential mechanisms of vibro-impact dynamics.

Internal resonance substantially modifies the linear dynamics. While it reduces the number of independent frequencies [13], it enlarges the set of grazing linear orbits. This motivates a focused study on grazing phenomena where classical non-resonant results no longer apply directly [1], [3].

In this work, we consider a particular class of grazing periodic linear orbits, denoted by $\mathrm{GLO}_c$, which is defined as the set of all convex combinations of the grazing linear modes previously identified in the non-resonant setting [7], [8]. We will show that, in the presence of internal resonance, a specific element of $\mathrm{GLO}_c$ generates a family of non-smooth modes, including 1IPP solutions, which were first identified in [9] and are closely related to modal structures observed in impact problems on elastic bars [14].

Furthermore, we prove that the dynamics in a neighborhood of a $\mathrm{GLO}_c$ exhibits a remarkably simple structure: no sticking phase occurs and the free-flight time remains almost constant near such orbits. This behavior sharply contrasts with the non-resonant case, where the free-flight time is unbounded and discontinuous near grazing [7]. Finally, we state a general result describing the dynamics of all orbits in this setting, which provides a unified framework for understanding grazing phenomena in vibro-impact systems with internal resonance.

# 2. Content

## 2.1. Model of 2-dof chain with a unilateral stop

We consider a 2-dof mass–spring chain in which the second mass is subject to a unilateral constraint (rigid stop) located at a distance $d > 0$, as illustrated in Figure 1. The governing equations are

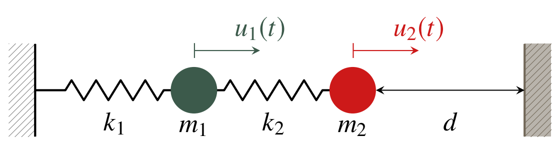


***Figure 1. The spring-mass chain system with unilateral contact condition***

$$\begin{cases} \mathbf{M}\ddot{\mathbf{u}} + \mathbf{K}\mathbf{u} = \mathbf{r} & \text{(2.1a)} \\ \mathbf{u}(0) = \mathbf{u}_0, \quad \dot{\mathbf{u}}(0) = \dot{\mathbf{u}}_0 & \text{(2.1b)} \\ u_2(t), \quad R(t) \leq 0, \quad (u_2(t) - d)R(t) = 0 & \text{(2.1c)} \\ \dot{\mathbf{u}}^+(t)^\top \mathbf{M}\dot{\mathbf{u}}^+(t) + \mathbf{u}^\top(t)\mathbf{K}\mathbf{u}(t) = \mathbf{E}(\mathbf{u}(t), \dot{\mathbf{u}}^+(t)) = \mathbf{E}(\mathbf{u}(0), \dot{\mathbf{u}}(0)) & \text{(2.1d)} \end{cases}$$

with

$$\mathbf{M} = (m_j)_{j=1}^2; \quad \mathbf{K} = (k_{ij})_{i,j=1}^2; \quad \mathbf{u}(t) = (u_j)_{j=1}^2; \quad \mathbf{r}(t) = (0, R(t))$$

where $\dot{u}_j$ and $\ddot{u}_j$ represent the velocity and acceleration of mass $j$, $j = 1, 2$, respectively. The model in question is a chain, for which $k_i$ is the stiffness of the spring $i$th and the stiffness matrix is

$$\mathbf{K} = \begin{pmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 \end{pmatrix}.$$

Condition (2.1c) says that mass $m_2$ is constrained on the right side by a rigid obstacle at a distance $d > 0$ from its equilibrium. There is only one constraint on mass $m_2$. The mass $m_1$ is not constrained in any way. The quantity $R(t)$ is the reaction force induced by the obstacle on mass $N$ at the time of gap closure. Generally, $R(t)$ is a measure. However, it is a Lipschitz function which is as regular as $\ddot{u}_2$ for solutions with sticking phases [15]. The present work targets non-dissipative dynamics and condition (2.1d) is enforced: the total energy of the system is preserved during the motion.

Matrices $\mathbf{M}$ and $\mathbf{K}$ are symmetric positive definite. Hence, there exists a matrix $\mathbf{P}$ of $\mathbf{M}$-orthogonal eigenmodes which diagonalizes both $\mathbf{M}$ and $\mathbf{K}$, that is $\mathbf{P}^\top \mathbf{M} \mathbf{P} = \mathbf{I}$ and $\mathbf{P}^\top \mathbf{K} \mathbf{P} = \mathbf{\Omega}^2 = (\omega_j^2)|_{j=1,2}$ where $\mathbf{I}$ is the $2 \times 2$ identity matrix; $\omega_j^2$ are the eigenfrequencies and $T_j$, the linear periods with $\omega_j T_j = 2\pi$, $j = 1, 2$.

***Internal resonance.*** The system is said to be in *internal resonance* if the frequency ratio is rational, i.e. $\omega_2/\omega_1 \in \mathbb{Q}$. Equivalently, there exist coprime integers $p, q \in \mathbb{N}$ such that $q\,\omega_2 = p\,\omega_1 \Leftrightarrow T_2 = \frac{p}{q} T_1$. Through out this paper, we focus on the special case $p = 2$, $q = 1$, which gives the 1:2 internal resonance condition

$$T_2 = 2T_1 \tag{2.2}$$

***Contact hyperplane as a section and free flight time.*** Let $\mathcal{H} = \{W = (\mathbf{u}, \dot{\mathbf{u}}^+) \in \mathbb{R}^4 : \ u_2 = d\}$ be the *contact hyperplane*. For $W \in \mathcal{H}$, let $\mathbf{u}(t; W)$ be the corresponding hybrid trajectory. Following a possible initial sticking episode, characterized by $\dot{u}_2 = 0$ over a finite time interval, the system evolves according to the free-flight dynamics until the next closing contact with the stop. The *free flight time* (FFT) to the contact hyperplane is,

$$\mathrm{FFT}(W) := \min\{t > 0 : \ u_2(t; W) = d\}, \tag{2.3}$$

where $t = 0$ is the departure time from $\mathcal{H}$. In other words, FFT denotes the return duration to $\mathcal{H}$. The notion of FFT used in this work should not be confused with the first return time (FRT) appearing in related studies [7], [8]. In contrast to FRT, which is defined with respect to a Poincaré section, FFT measures only the time interval over which the system evolves under free-flight dynamics. In the non resonance case, FRT can be very large and may even fail to exist [7], [8]. Here, with the $1:2$ resonance, we will show that FFT always exists. Moreover $\mathrm{FFT} \le T_2$. We now define a class of special orbits, namely, grazing linear orbits on $\mathcal{H}$ with linear dynamics. These orbits are well known to trigger grazing bifurcations [3], [5]. Due to internal resonance, they have a period $T_2$ and are more numerous than in the absence of internal resonance.

**Definition 2.1** (Grazing linear orbit (GLO))**.** *A* grazing linear orbit *(GLO) is a* periodic *orbit of the free-flight linear system that reaches the contact hyperplane $\mathcal{H}$ only at a grazing event.*

After a time shift we often assume that a grazing instant is at $t = 0$, so that $u_2(0) = d, \dot{u}_2(0) = 0$. In particular, two pure-mode grazing linear orbits are fundamental to our analysis.

**Definition 2.2** (Grazing linear orbit modes ($\mathrm{GLOM}_1$, $\mathrm{GLOM}_2$))**.** *For $i \in \{1, 2\}$, the* grazing linear orbit mode $\mathrm{GLOM}_i$ *is the pure-mode periodic solution*

$$\mathbf{u}^i(t) = A_i\,\mathbf{P}\mathbf{e}_i \cos(\omega_i t), \qquad A_i := \frac{d}{P_{2i}},$$

*so that $u_2^i(0) = d$ and $\dot{u}_2^i(0) = 0$, and $u_2^i(t) < d$ for all $0 < t < T_i$, where $\mathbf{P}$ is the matrix that diagonalizes both $\mathbf{M}$ and $\mathbf{K}$.*

We now extend the notion of GLOMs to include their convex combinations.

**Definition 2.3** ($\mathrm{GLO}_c$ - Convex combinations of GLOMs)**.** *Denote by $W_0^1$ and $W_0^2$ the data on the contact hyperplane $\mathcal{H}$ of $\mathrm{GLOM}_1$ and $\mathrm{GLOM}_2$. The set of all convex combinations $W^\lambda$ of $W_0^1$ and $W_0^2$ corresponding to a unique grazing linear orbit represents all $\mathrm{GLO}_c$,*

$$W_0^\lambda = (1-\lambda)W_0^1 + \lambda W_0^2 = \Big((1-\lambda)\,d\frac{P_{11}}{P_{21}} + \lambda\,d\frac{P_{12}}{P_{22}}, d, 0, 0\Big), \qquad \lambda \in [0, 1]. \tag{2.4}$$

We also write $\{\mathrm{GLO}_c\} := [\mathrm{GLOM}_1, \mathrm{GLOM}_2]$ for this family.

**Remark 2.1.** *In the 2-dof case, the grazing-start constraints $u_2(0) = d$ and $\dot{u}_2(0) = 0$ define a two-dimensional affine plane in $\Sigma$, which may be parameterized by $(u_1(0), v_1(0))$. The set of $\mathrm{GLO}_c$ is a one-dimensional subset of this plane.*

Finally, an important family of nonsmooth periodic orbits, previously studied in [9], is closely related to grazing bifurcations and the family $\mathrm{GLO}_c$, as will be shown.

**Definition 2.4** (One-impact-per-period (1IPP) orbit)**.** *A* one-impact-per-period *(1IPP) orbit is a $T$-periodic hybrid trajectory whose* minimal *period is $T$ and that undergoes exactly one closing impact with the unilateral constraint during each period [9].*

## 2.2. Main results

We consider the 2-dof unilaterally constrained chain governed by (2.1a)–(2.1d) and the internal resonance condition (2.2). First, the central role of $\mathrm{GLO}_c$ is highlighted since one of them generates an invariant manifold of non-smooth modes as for a bar [14], [16].

**Theorem 2.1** (Nonlinear modes with one impact-per-period are generated by a $\mathrm{GLO}_c$)**.** *Let $W^T$ be the data on $\mathcal{H}$ of a 1IPP nonlinear mode with period $T$, for $T \notin T_1\mathbb{N} \cup T_2\mathbb{N}$. Then, there exists a unique $\mathrm{GLO}_c$, that is $\lambda \in (0, 1]$, such that*

$$W^T \longrightarrow W^\lambda \qquad \textit{as} \qquad T \to T_2. \tag{2.5}$$

Second, the dynamics in a neighborhood of a $\mathrm{GLO}_c$ is shown to be particularly simple.

**Theorem 2.2** (Near a $\mathrm{GLO}_c$)**.** *The FFT near a $\mathrm{GLO}_c \neq \mathrm{GLOM}_1$ is close to and less than $T_2$.*

The above result follows from the simplicity of the underlying linear dynamics. Owing to internal resonance, linear solutions are $T_2$-periodic, and $T_2$ is the minimal period for all solutions except those associated with mode 1. In the same spirit, the following theorem about zero or infinitely many contacts is specific to the resonant case, which is different from the non-internal resonance setting [8], where trajectories may exhibit zero, one, or infinitely many contacts.

**Theorem 2.3** (Zero or infinitely many contacts)**.** *There are only two kinds of orbits:*

*(i)* No contact*: $u_2(t) < d$ for all $t \in \mathbb{R}$.*

*(ii)* Infinitely many contacts*: the set of all contact times $CT = \{t \in \mathbb{R}, u_2(t) = d\}$ is infinite, $\inf CT = -\infty$, $\sup CT = +\infty$ and between two consecutive contacts $\mathrm{FFT} \leq T_2$.*
*Moreover, $\mathrm{FFT} = T_2$ if and only if the solution is a GLO different from $\mathrm{GLOM}_1$.*

In particular, for any nonlinear solution, the free flight time is strictly less than $T_2$.

## 2.3. The set of $\mathrm{GLO}_c$

Taking a convex combination of GLOMs yields a grazing linear orbit (GLO). This fact is straightforward, and we briefly explain it here. Let

$$\mathbf{u}(t) = (1-\lambda)\mathbf{u}^1(t) + \lambda\mathbf{u}^2(t), \qquad 0 < \lambda < 1,$$

be a convex combination of the two GLOMs. Since both $\mathbf{u}^1$ and $\mathbf{u}^2$ are solutions of the linear system with period $T_2$, their convex combination $\mathbf{u}$ is also a $T_2$-periodic solution of the linear system. Moreover, for each $i \in \{1, 2\}$, the displacement of the second mass along the $i$th GLOM is given by $u_2^i(t) = d\cos(\omega_i t)$. Consequently,

$$u_2(t) = (1-\lambda)u_2^1(t) + \lambda u_2^2(t) \leq d \quad \text{for all } t.$$

Thus, $u$ satisfies the unilateral constraint $u_2 \leq d$ at all times and is therefore a solution of the vibro-impact system. Furthermore, this solution grazes the contact hyperplane $\mathcal{H}$, since $u_2(0) = d$ and $\dot{u}_2(0) = 0$.

With internal resonance condition, linear combinations of GLOMs that reach the wall simultaneously give rise to GLOs, leading to a strictly larger set of GLOs than the set of all $\mathrm{GLO}_c$. For $1 < \lambda \leq 4/3$, the corresponding solutions form a nontrivial family of GLO that do not belong to $\{\mathrm{GLO}_c\}$.

**Lemma 2.1.** $\max\left((1-\lambda)\cos(2t) + \lambda\cos(t)\right) = 1$ *if and only if* $\lambda \in [0, 4/3]$.

*Proof.* Let $f(t) = \cos(2t) = 2\cos^2 t - 1$ and $g(t) = \cos t$. Both functions are $2\pi$-periodic and attain their maximum value 1 at $t = 0$. Consider the convex combination $h(t) = (1-\lambda)f(t) + \lambda g(t)$ and write it as a function of $X = g(t) \in [-1, 1]$: $h(X) = (1-\lambda)(2X^2 - 1) + \lambda X$.

- If $0 \leq \lambda \leq 1$, then $h(X) \leq (1-\lambda) + \lambda = 1$, with equality at $X = 1$, hence $\max h = 1$.
- If $\lambda < 0$, taking $X = -1$ yields $h(-1) = 1 + 2|\lambda| > 1$.
- If $1 < \lambda \leq 4/3$, writing $\lambda = 1 + \mu$ with $0 < \mu \leq 1/3$ gives $h(X) = \mu + (1+\mu)X - 2\mu X^2$, which is strictly increasing on $[-1, 1]$, so $h(X) \leq h(1) = 1$.
- If $\lambda > 4/3$, then $h(0) = 1$, $h'(0) = 0$, $h''(0) = 3\lambda - 4 > 0$, so $h(t) > 1$ for $t \neq 0$ small enough.

Therefore, $\max h = 1$ if and only if $\lambda \in [0, 4/3]$. ☐

## 2.4. Continuity of 1IPP to a $\mathrm{GLO}_c$

In this section, we will show that there is a branch of the 1IPP orbits emerges from a GLO, that is Theorem 2.1. First, let us recall the definition of the nonlinear solution with 1-IPP [9] and define the initial data for the set of grazing linear orbits which are proven to be the limit of the 1-IPPs.

**Initial data for 1-IPP.** Denote the post-impact state as $W = (\mathbf{u}, \dot{\mathbf{u}}^+) \in \mathbb{R}^4$ with $u_2 = d$ on $\mathcal{H}$. For $T \notin T_1\mathbb{N} \cup T_2\mathbb{N}$, the unique 1-IPP orbit of period $T$ (Definition 2.4, cf. [9]) is characterized by the post-impact initial data

$$W^T := \big(\mathbf{u}(0;T), \dot{\mathbf{u}}^+(0;T)\big), \qquad \mathbf{u}(0;T) = \dot{u}_2^+(0;T)\,\mathbf{w}(T), \tag{2.6}$$

$$\dot{\mathbf{u}}^+(0;T) = \dot{u}_2^+(0;T)\,\mathbf{e}_2, \qquad \dot{u}_2^+(0;T) = \frac{d}{w_2(T)}, \tag{2.7}$$

$$w_k(T) = \sum_{j=1}^{2} a_{kj}\,\Phi_j(T), \quad a_{kj} = P_{kj}P_{j2}^{-1}, \quad \Phi_j(t) = \frac{\sin(\omega_j t)}{\omega_j\big(1 - \cos(\omega_j t)\big)}. \tag{2.8}$$

From an analytical viewpoint, these expressions reflect the structure of the underlying linear dynamics. In particular, the quantity $w(T)$ can be regarded as a vector-valued holomorphic function of $T$, whose poles are associated with the linear periods of the system. This structure explains the form of the formulas. We also emphasize that such closed-form expressions are specific to this setting, and do not extend in a straightforward way to multi-impact ($k$-IPP).

**Grazing limit and resonant GLO data.** The associated GLO is the $T_2$-periodic solution whose data on $\mathcal{H}$ is

$$W_0 := \big(\mathbf{u}(0;T_2), \mathbf{0}\big) \in \mathcal{H}, \qquad u_k(0;T_2) = d\,\frac{\sum_{j\in I} a_{kj}\omega_j^{-2}}{\sum_{j\in I} a_j\omega_j^{-2}}, \quad k=1,2, \tag{2.9}$$

where $I \subset \{1,2\}$ is the set of resonant modes satisfying $T_2 \in T_j\mathbb{N}$ and $a_j := a_{2j} = P_{2j}P_{j2}^{-1}$. Theorem 2.1 can now be proven as follows.

*Proof.* Fix $T \notin T_1\mathbb{N} \cup T_2\mathbb{N}$ sufficiently close to $T_2$. By [9], the corresponding 1IPP orbit of period $T$ is uniquely determined by the post-impact initial data $W^T = (\mathbf{u}(0;T), \dot{\mathbf{u}}^+(0;T)) \in \mathcal{H}$ given by (2.6)–(2.7).

Let $I \subset \{1,2\}$ be the set of resonant indices, i.e. those $j$ such that $T_2 \in T_j\mathbb{N}$. For $j \in I$, a Taylor expansion about $T_2$ yields

$$\sin(\omega_j T) = \omega_j(T-T_2) + O\big((T-T_2)^3\big), \qquad 1-\cos(\omega_j T) = \frac{\omega_j^2}{2}(T-T_2)^2 + O\big((T-T_2)^4\big),$$

and hence

$$\Phi_j(T) = \frac{\sin(\omega_j T)}{\omega_j\big(1-\cos(\omega_j T)\big)} = \frac{2}{\omega_j^2\,(T-T_2)} + O(1), \qquad T \to T_2. \tag{2.10}$$

For $j \notin I$, the term $1-\cos(\omega_j T)$ stays bounded away from $0$ near $T_2$, so $\Phi_j(T) = O(1)$ as $T \to T_2$.

Using (2.10) and the boundedness of the non-resonant terms, we obtain for $k = 1,2$,

$$w_k(T) = \sum_{j=1}^{2} a_{kj}\Phi_j(T) = \frac{2}{T-T_2}\sum_{j\in I} a_{kj}\omega_j^{-2} + O(1), \qquad T \to T_2. \tag{2.11}$$

In particular,

$$w_2(T) = \frac{2}{T-T_2}\sum_{j\in I} a_{2j}\omega_j^{-2} + O(1).$$

Since $\sum_{j\in I} a_{2j}\omega_j^{-2} \neq 0$ [9], it follows that $|w_2(T)| \to \infty$ as $T \to T_2$ and therefore $\dot{u}_2^+(0;T) = \frac{d}{w_2(T)} \longrightarrow 0$ as $T \to T_2$, which proves $\dot{\mathbf{u}}^+(0;T) \to \mathbf{0}$.

Finally, since $\mathbf{u}(0;T) = \dot{u}_2^+(0;T)\mathbf{w}(T)$, we may write componentwise

$$u_k(0;T) = \dot{u}_2^+(0;T)\, w_k(T) = d\,\frac{w_k(T)}{w_2(T)}, \qquad k = 1,2.$$

A combination of this with (2.11) gives

$$u_k(0;T) \longrightarrow d\,\frac{\displaystyle\sum_{j\in I} a_{kj}\omega_j^{-2}}{\displaystyle\sum_{j\in I} a_{2j}\omega_j^{-2}} =: u_k(0;T_2), \qquad k = 1,2.$$

Therefore, $W^T \to W_0 := (\mathbf{u}(0;T_2), \mathbf{0})$ as $T \to T_2$. This proves (2.5). □

The last step is to show that $W_0$ corresponds to a $\mathrm{GLO}_c$.

**Lemma 2.2** (Explicit convex combinations)**.** *Let* $W_0$ *be given by* (2.9)*. Then, there exists a* $\lambda \in (0,1]$ *such that* $W_0 = (1-\lambda)W_0^1 + \lambda W_0^2$.

*Proof.* With $I = \{1,2\}$, formula (2.9) yields

$$u_1(0;T_0) = d\,\frac{a_{11}\omega_1^{-2} + a_{12}\omega_2^{-2}}{a_{21}\omega_1^{-2} + a_{22}\omega_2^{-2}}, \qquad u_2(0;T_0) = d, \qquad \dot{u}(0;T_0) = 0,$$

where $a_{kj} = P_{kj}P_{j2}^{-1}$. Since $\dfrac{a_{1j}}{a_{2j}} = \dfrac{P_{1j}P_{j2}^{-1}}{P_{2j}P_{j2}^{-1}} = \dfrac{P_{1j}}{P_{2j}}, \qquad j = 1,2$, we may rewrite

$$u_1(0;T_0) = (1-\lambda)\, d\frac{P_{11}}{P_{21}} + \lambda\, d\frac{P_{12}}{P_{22}}, \qquad \lambda := \frac{a_{22}\omega_2^{-2}}{a_{21}\omega_1^{-2} + a_{22}\omega_2^{-2}}.$$

Because $\omega_j^{-2} > 0$ and $a_{2j} > 0$ for $j = 1,2$, we have $\lambda \in [0,1]$. The identity for $W_0$ follows since $u_2(0;T_0) = d$ and $\dot{u}(0;T_0) = 0$ for $W_0^1$ and $W_0^2$, then for $W_0$. □

## 2.5. Zero or an infinite number of contacts and $\mathrm{FFT} \le T_2$

This section completes the proof of Theorem 2.3 which will be used to prove Theorem 2.2. We show that trajectories either experience no contact or infinitely many contacts, and that whenever a contact occurs the associated FFT satisfies $\mathrm{FFT} \le T_2$. If no contact occurs, the motion is governed entirely by the linear dynamics and is therefore $T_2$-periodic. We henceforth assume that a contact occurs. By invariance under time translation, we can assume that a contact occurs at $t = 0$. Denote by $\mathbf{u}^{\#}$ the solution of the unconstrained linear system whose position and velocity coincide with those of the constrained trajectory $\mathbf{u}$ at $t = 0^+$, immediately after detachment. Then $\mathbf{u}(t) = \mathbf{u}^{\#}(t)$ throughout the free-flight time. If a sticking phase occurs, it has finite duration for the chain [15]; in that case, we redefine $t = 0$ as the end of the sticking phase so that the motion again starts with free flight. The linear trajectory $\mathbf{u}^{\#}$ is $T_2$-periodic, although $\mathbf{u}$ need not be periodic. Since $u_2^{\#}(T_2) = d$, the constrained trajectory must experience a contact no later than time $T_2$, which implies

$$\mathrm{FFT} \leq T_2.$$

If the contact occurs with nonzero velocity, i.e. $u_2(0^-) > 0$, then $u_2^{\#}(0) = -u_2(0^-) < 0$, and hence $u_2^{\#}(-\tau) > d$ for $\tau > 0$ sufficiently small. By $T_2$-periodicity of $u^{\#}$, this yields $u_2^{\#}(T_2 - \tau) > d$, implying that a new contact occurs strictly before $T_2$. Thus, for a true impact,

$$\mathrm{FFT} < T_2. \tag{2.12}$$

For a grazing contact, $\mathrm{FFT} = T_2$ is possible but not mandatory. Repeating the above argument after each detachment shows that once a contact occurs, the trajectory necessarily undergoes infinitely many contacts. Moreover, the number of contacts (when entering or exiting $\mathcal{H}$, a sticking counts as two, an impact and a grazing count as one) is finite on any compact sets [17]. This completes the proof of Theorem 2.3.

## 2.6. Simplified dynamics near a $\mathrm{GLO}_c$

In this section, we prove Theorem 2.2. We will show that, in a neighborhood of a $\mathrm{GLO}_c$ distinct from $\mathrm{GLOM}_1$, the dynamics simplifies and the associated free flight time is $\leq T_2$. A key preliminary step is to exclude the occurrence of micro-contacts near $\mathrm{GLO}_c$. Here, a micro-contact refers to an additional contact occurring close to the contact at $t = 0$ for trajectories with initial data on $\mathcal{H}$.

**Proposition 2.1.** *There is no micro-contact in the vicinity of a* $\mathrm{GLO}_c$ *which is not a GLOM.*

The proof follows a similar strategy to the non-resonance case in Section 4.2, [7], but we provide the full argument here for completeness.

*Proof.* Let $W_0 = W_0(\lambda)$ be a $\mathrm{GLO}_c$ with $\lambda \in (0, 1)$. Then, no micro-contact means that there exist $\varepsilon > 0$, $\eta > 0$, and $\delta > 0$ such that

$$u_2(t; W_0) \leq d - \eta, \qquad \forall\, t \in [\varepsilon,\, T_2 - \varepsilon], \tag{2.13}$$

and, for every $W \in \mathcal{H} \cap B_\delta(W_0)$,

$$u_2(t; W) < d, \qquad \forall\, t \in [\varepsilon,\, T_2 - \varepsilon]. \tag{2.14}$$

Consequently, any contact time after $t = 0$, $FFT(W)$ belongs to $(T_2 - \varepsilon,\, T_2]$. In particular, there is no micro-contact on $(0,\, T_2 - \varepsilon]$ for all such nearby initial data. Since

$W_0$ is associated to a $\mathrm{GLO}_c$ and the free-flight dynamics is linear, the second component along the free-flight trajectory issued from $W_0(\lambda)$ can be written as

$$u_2(t; W_0) = d\big((1-\lambda)\cos(\omega_1 t) + \lambda\cos(\omega_2 t)\big), \qquad \omega_1 = 2\omega_2.$$

Hence $u_2(t; W_0) \leq d$ for all $t$, and equality $u_2(t; W_0) = d$ can occur only if $\cos(\omega_1 t) = \cos(\omega_2 t) = 1$. Under $\omega_1 = 2\omega_2$ this implies $t \in T_2\mathbb{Z}$. Therefore, $u_2(t; W_0) < d, \forall\, t \in (0, T_2)$. Fix any $\varepsilon \in (0, T_2/2)$. By continuity of $t \mapsto u_2(t; W_0)$ and compactness of $[\varepsilon, T_2 - \varepsilon]$, the maximum

$$m_\varepsilon := \max_{t\in[\varepsilon, T_2-\varepsilon]} u_2(t; W_0)$$

is attained and satisfies $m_\varepsilon < d$. Define $\eta := d - m_\varepsilon > 0$. This proves (2.13). Next, since the free-flight solution depends continuously on the initial data, there exists $C > 0$ such that

$$\sup_{t\in[\varepsilon, T_2-\varepsilon]} \big|u_2(t; W) - u_2(t; W_0)\big| \leq C\|W - W_0\| \qquad \text{for all } W \in \mathbb{R}^4.$$

Choose $\delta := \eta/(2C)$. Then, for any $W \in B_\delta(W_0)$ and any $t \in [\varepsilon, T_2 - \varepsilon]$,

$$u_2(t; W) \leq u_2(t; W_0) + \frac{\eta}{2} \leq (d - \eta) + \frac{\eta}{2} = d - \frac{\eta}{2} < d,$$

which is (2.14). Finally, if a contact occurs at some time $\tau \in (0, T_2]$, then necessarily $u_2(\tau; W) = d$. By (2.14), such a $\tau$ cannot lie in $[\varepsilon, T_2 - \varepsilon]$. Thus any contact time after $t = 0$ must belong to $(T_2 - \varepsilon, T_2]$, which yields the claimed bound on the FFT and excludes micro-contacts on $(0, T_2 - \varepsilon]$. □

The same proof is valid for $\mathrm{GLOM}_2$. For $\mathrm{GLOM}_1$ the situation is slightly more complicated since FFT can be also near $T_1$. We can now prove theorem 2.2.

*Proof.* By Theorem 3, we have $\mathrm{FFT} \leq T_2$. Consider initial data $W$ in a neighborhood of $W^\lambda$ such that $\mathrm{GLO}_c \neq \mathrm{GLOM}_1$. The associated linear dynamics is exactly $T_2$-periodic. We restrict to a neighborhood of $W^\lambda$ that does not intersect $\mathrm{GLOM}_1$.

Taking the unilateral contact into account, let $X(t; W)$ and $X(t; W^\lambda)$ denote the corresponding free-flight trajectories issued from $W$ and $W^\lambda$, respectively. Since, away from contact, the dynamics is linear and conservative, the mechanical energy

$$\mathbf{E}(W) = \tfrac{1}{2}\big(\dot{\mathbf{u}}^\top \mathbf{M}\dot{\mathbf{u}} + \mathbf{u}^\top \mathbf{K}\mathbf{u}\big)$$

is preserved. This induces a norm under which the flow is isometric. In particular, for all $t \in [0, T_2]$,

$$\|X(t; W) - X(t; W^\lambda)\| = \|W - W^\lambda\|.$$

Hence, for $W$ sufficiently close to $W^\lambda$, the corresponding trajectory remains uniformly close to the linear orbit on $[0, T_2]$.

By Proposition 2.1, the trajectory completes almost one full loop before the next impact, and therefore $FFT \approx T_2$. □

## 3. Conclusions

We have presented a concise analysis of a 2-dof vibro-impact oscillator with internal resonance, focusing on a mass–spring chain and a distinguished subset of grazing linear orbits. We have shown that the 1IPP family is generated by a special $\mathrm{GLO}_c$, and established the continuity of FFT, in a neighborhood of this orbit. These results clarify the key mechanisms governing the grazing limit in the presence of internal resonance.

The extension of the present analysis to other resonance ratios and higher-dimensional systems remains an open direction. In the case of $1 : n$ resonances, one may expect similar mechanisms to persist, although the analysis would require additional technical developments. For general $m : n$ resonances with $m, n \neq 1$, the interaction between modes becomes more intricate and the structure of grazing orbits is less explicit. In multi-degree-of-freedom systems, further difficulties arise due to the coexistence of commensurate and incommensurate frequencies, for instance, one may have relations such as $T_2 = 2T_1$ while $T_1/T_3$ is irrational, leading to a richer dynamical structure.

A complete stability analysis of the $\mathrm{GLO}_c$ and a systematic numerical investigation of these extended settings lie beyond the scope of the present short note and will be addressed in forthcoming work in a more general framework.

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