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ON BALANCING NUMBERS BY THE MATRIX METHOD

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Abstract. In this study, we define a balancing B-matrix which is similar to the Fibonacci Q-matrix and the Lucas Q_L-matrix. Using this matrix representation, we obtain some well-known equalities and a Binet-like formula for balancing numbers. Moreover, some arithmetic properties of balancing numbers are established.
Keywords: Balancing numbers, matrix methods, matrix representation.

1. Introduction

Fibonacci and Lucas numbers with their generalizations have many interesting properties and applications to various fields of science and art. For the prettiness and richness applications of these numbers and their relatives to science and nature one can see [1], [2]. Consider the following 2×2 matrix

$$Q = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix},$$

called the Q-matrix, which was first studied by King [3] (see also [2, pp. 362]). Then for a positive integer n, Q^n has the form

$$Q^n = \begin{pmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{pmatrix},$$

where F_n is the nth Fibonacci number. We have

$$(-1)^n = \det(Q^n) = F_{n-1}F_{n+1} - F_n^2,$$

so we obtain Cassini's formula for the Fibonacci numbers.

In 2010, Köken and Bozkurt [4] considered the Lucas Q_L-matrix defined by

$$Q_L = \begin{pmatrix} 3 & 1 \\ 1 & 2 \end{pmatrix}$$

then, for a positive integer n, we have

$$\begin{pmatrix} L_{n+1} \\ L_n \end{pmatrix} = Q_L \begin{pmatrix} F_n \\ F_{n-1} \end{pmatrix} \quad \text{and} \quad 5 \begin{pmatrix} F_{n+1} \\ F_n \end{pmatrix} = Q_L \begin{pmatrix} L_n \\ L_{n-1} \end{pmatrix},$$

where L_n is the nth Lucas number. They use this matrix representation to find some well-known equalities and a Binet-like formula for the Fibonacci and Lucas numbers.

Consider the Pell numbers P_n and Pell-Lucas numbers Q_n (also known as the *Companion Pell*) which are defined by the recurrence relations

$$\begin{aligned} P_0 &= 0, \quad P_1 = 1, \quad P_n = 2P_{n-1} + P_{n-2}, \quad n \geq 2, \\ Q_0 &= 1, \quad Q_1 = 1, \quad Q_n = 2Q_{n-1} + Q_{n-2}, \quad n \geq 2. \end{aligned}$$

It is important to note that Pell and Pell–Lucas numbers serve as a bridge linking number theory, combinatorics, graph theory, geometry, trigonometry, and analysis. These numbers occur, for example, in the study of lattice walks, and the tilings of linear and circular boards using unit square tiles and dominoes [5].

Similar to the case of Fibonacci and Lucas numbers, Koshy [5] also use matrices to generate Pell numbers and Pell-Lucas numbers by defining the matrix P and Q respectively as

$$P = \begin{pmatrix} 2 & 1 \\ 1 & 0 \end{pmatrix} \text{ and } Q = \begin{pmatrix} 3 & 1 \\ 1 & 0 \end{pmatrix}$$

then,

$$P^n = \begin{pmatrix} P_{n+1} & P_n \\ P_n & P_{n-1} \end{pmatrix},$$

where P_n is the n-th Pell number and

$$Q^n = \begin{cases} 2^{n/2} \begin{pmatrix} P_{n+1} & P_n \\ P_n & P_{n-1} \end{pmatrix} & \text{if } n \text{ is even,} \\ 2^{\lfloor n/2 \rfloor} \begin{pmatrix} Q_{n+1} & Q_n \\ Q_n & Q_{n-1} \end{pmatrix} & \text{otherwise,} \end{cases}$$

where Q_n is the n-th Pell number.

In this paper, we also use matrix representation for the balancing sequence $\{B_n\}_{n=0}^{\infty}$ which is defined by the recursion

$$B_0 = 0, \quad B_1 = 1, \quad B_n = 6B_{n-1} - B_{n-2}, \quad n \geq 2$$

to find some well-known equalities, a Binet-like formula and some arithmetic properties of balancing numbers. Note that balancing numbers are quite well-known and closely connected with triangular numbers, Pells numbers and Fibonacci numbers [6], [7].

2. Matrix representation of balancing numbers

Consider the matrix

$$B = \begin{pmatrix} 6 & 1 \\ -1 & 0 \end{pmatrix},$$

we have

$$B^2 = \begin{pmatrix} 35 & 6 \\ -6 & -1 \end{pmatrix} = \begin{pmatrix} B_3 & B_2 \\ -B_2 & -B_1 \end{pmatrix} \text{ and } B^3 = \begin{pmatrix} B_4 & B_3 \\ -B_3 & -B_2 \end{pmatrix}.$$

More generally, we have the following result.

Theorem 2.1. *Let n be a positive integer. Then*

$$B^n = \begin{pmatrix} B_{n+1} & B_n \\ -B_n & -B_{n-1} \end{pmatrix},$$

where B_n is the nth balancing number.

Proof. We will prove it by induction. The result is clearly true when $n = 1$. Now assume that it is true for an arbitrary integer $k \geq 2$

$$B^k = \begin{pmatrix} B_{k+1} & B_k \\ -B_k & -B_{k-1} \end{pmatrix}.$$

Then, using the balancing recurrence, we have

$$\begin{aligned} B^{k+1} &= B^k \cdot B \\ &= \begin{pmatrix} B_{k+1} & B_k \\ -B_k & -B_{k-1} \end{pmatrix} \begin{pmatrix} 6 & 1 \\ -1 & 0 \end{pmatrix} \\ &= \begin{pmatrix} 6P_{k+1} - B_k & B_{k+1} \\ -6B_k + B_{k-1} & -B_k \end{pmatrix} \\ &= \begin{pmatrix} B_{k+2} & B_{k+1} \\ -B_{k+1} & -B_k \end{pmatrix}. \end{aligned}$$

So the result is true when $n = k + 1$. Thus, by induction, the assertion is true for every integer $n \geq 1$. □

An immediate consequence of Theorem 2.1 is the Cassini-like formula for B_n.

Corollary 2.1. *Let n be a positive integer. Then,*

$$B_{n+1}B_{n-1} - B_n^2 = -1.$$

Proof. Denote by $\det(M)$ the determinant of matrix M. Then,

$$(\det(B))^n = \det(B^n) = \det\begin{pmatrix} B_{n+1} & B_n \\ -B_n & -B_{n-1} \end{pmatrix} = -B_{n+1}B_{n-1} + B_n^2.$$

But $\det(B) = 1$ so $\det(B^n) = 1$. Thus, $B_{n+1}B_{n-1} - B_n^2 = -1$ as desired. □

Theorem 2.2. *Let n be a positive integer. Then, a Binet-like formula for the balancing number is*

$$B_n = \frac{\lambda_1^n - \lambda_2^n}{4\sqrt{2}},$$

where $\lambda_1 = 3 + 2\sqrt{2}, \quad \lambda_2 = 3 - 2\sqrt{2}$ are two roots of the characteristic equation of the balancing recurrence $\lambda^2 - 6\lambda + 1 = 0$.

Proof. We can write the characteristic equation of the matrix B as $\det(B - \lambda I) = 0$ i.e

$$\lambda^2 - 6\lambda + 1 = 0.$$

If we calculate the eigenvalues and eigenvectors of the matrix B, we obtain

$$\lambda_1 = 3 + 2\sqrt{2}, \quad \lambda_2 = 3 - 2\sqrt{2}$$

and the matrix

$$U = \begin{pmatrix} -3 - 2\sqrt{2} & -3 + 2\sqrt{2} \\ 1 & 1 \end{pmatrix} \quad \text{and} \quad U^{-1} = \begin{pmatrix} -\dfrac{\sqrt{2}}{8} & \dfrac{-3\sqrt{2} + 4}{8} \\ \dfrac{\sqrt{2}}{8} & \dfrac{3\sqrt{2} + 4}{8} \end{pmatrix}$$

then,

$$B = U \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} U^{-1}.$$

Therefore,

$$B^n = U \begin{pmatrix} \lambda_1^n & 0 \\ 0 & \lambda_2^n \end{pmatrix} U^{-1} = \frac{1}{4\sqrt{2}} \begin{pmatrix} \lambda_1^{n+1} - \lambda_2^{n+1} & \lambda_1^n - \lambda_2^n \\ -\lambda_1^n + \lambda_2^n & -\lambda_1^{n-1} + \lambda_2^{n-1} \end{pmatrix}.$$

By Theorem 2.1 and by equating the lower left-hand elements from both sides, we obtain

$$B_n = \frac{\lambda_1^n - \lambda_2^n}{4\sqrt{2}}.$$

□

Theorem 2.3. *Let m, n be the positive integers m, n. Then,*

$$B_{n+m} = B_{n+1}B_m - B_nB_{m-1}.$$

Proof. Since $B^{m+n} = B^m B^n$, by Theorem 2.1, we have

$$\begin{pmatrix} B_{m+n+1} & B_{m+n} \\ -B_{m+n} & -B_{m+n-1} \end{pmatrix} = \begin{pmatrix} B_{m+1} & B_m \\ -B_m & -B_{m-1} \end{pmatrix} \begin{pmatrix} B_{n+1} & B_n \\ -B_n & -B_{n-1} \end{pmatrix}$$
$$= \begin{pmatrix} B_{m+1}B_{n+1} - B_m B_n & B_{m+1}B_n - B_m B_{n-1} \\ -B_m B_{n+1} + B_{m-1}B_n & -B_m P_n + B_{m-1}B_{n-1} \end{pmatrix}.$$

The formula follows by equating the lower left-hand elements from both sides. □

Corollary 2.2. *For all positive integers n, the following equalities are valid.*

(i) $B_{2n} = B_{n+1}B_n - B_n B_{n-1}$.

(ii) $B_{2n-1} = B_n^2 - B_{n-1}^2$.

(iii) $B_n^2 = B_1 + B_3 + \cdots + B_{2n-1}$.

Proof. (i) Let $m = n$ in Theorem 2.3, we have

$$B_{2n} = B_{n+1}B_n - B_n B_{n-1}.$$

(ii) We have

$$B_{2n-1} = B_{n-1+n} = B_n \cdot B_n - B_{n-1} \cdot B_{n-1} \text{ (by Theorem 2.3)}$$
$$= B_n^2 - B_{n-1}^2.$$

(iii) Since the formula in (ii), we have

$$B_1 + B_3 + \cdots + B_{2n-1} = B_1^2 - B_0^2 + B_2^2 - B_1^2 + \cdots + B_n^2 - B_{n-1}^2 = B_n^2.$$

□

Since $\det(B) = 1$, the matrix B is invertible, so is B^m. Thus, the inverse B^{-m} is given by

$$B^{-m} = \frac{1}{\det(B^m)} \begin{pmatrix} -B_{m-1} & -B_m \\ B_m & B_{m+1} \end{pmatrix} = \begin{pmatrix} -B_{m-1} & -B_m \\ B_m & B_{m+1} \end{pmatrix}$$

Thus,

$$B^{n-m} = B^n \cdot B^{-m} = \begin{pmatrix} B_{n+1} & B_n \\ -B_n & -B_{n-1} \end{pmatrix} \begin{pmatrix} -B_{m-1} & -B_m \\ B_m & B_{m+1} \end{pmatrix}.$$

So,

$$\begin{pmatrix} B_{n-m+1} & B_{n-m} \\ -B_{n-m} & -B_{n-m-1} \end{pmatrix} = \begin{pmatrix} -B_{n+1}B_{m-1} + B_n B_m & -B_{n+1}B_m + B_n B_{m+1} \\ B_n B_{m-1} - B_{n-1}B_m & B_n B_m - B_{n-1}B_{m+1} \end{pmatrix}$$

This implies that

$$B_{n-m} = B_n B_{m+1} - B_{n+1} B_m \tag{2.1}$$

for all positive integers $n \geq m$.

Note that if we extend the balancing number B_m to integer indices by setting $B_{-m} = -B_m$, then the formula (2.1) can be obtained from Theorem 2.3 by changing m for $-m$.

Now, we find some interesting arithmetic properties of balancing numbers.

Lemma 2.1. *Let m be a positive integer. Then* $\gcd(B_m, B_{m-1}) = 1$.

Proof. We have

$$\gcd(B_k, B_{k-1}) = \gcd(6B_{k-1} - B_{k-2}, B_{k-1}) = \gcd(B_{k-1}, B_{k-2}),$$

for all $k \geq 2$. Thus, $\gcd(B_m, B_{m-1}) = \gcd(B_1, B_0) = 1$, as desired. □

Theorem 2.4. *Let m, n be positive integers. Then,*

(i) $B_n \mid B_{mn}$.

(ii) *if* $B_n \mid B_m$ *then* $n \mid m$.

Proof. (i) We prove this statement by induction on m. Since the statement is true for $m = 1$, assume it is true for an arbitrary integer $m \geq 1$, then by Theorem 2.3, we have

$$\begin{aligned} B_{n(m+1)} &= B_{n+nm} \\ &= B_{n+1} B_{nm} - B_n B_{nm-1}. \end{aligned}$$

Since $B_n \mid B_{nm}$, by the inductive hypothesis, it follows that $B_n \mid B_{n(m+1)}$. Thus, by induction, $B_n \mid B_{mn}$ for all integers $m \geq 1$.

(ii) By the division algorithm, let $m = nk + r$, where $0 \leq r < n$. Then, by Theorem 2.3, we have

$$\begin{aligned} B_m &= B_{nk+r} \\ &= B_{nk} B_{r+1} - B_{nk-1} B_r. \end{aligned}$$

Since $B_n \mid B_m$ and $B_n \mid B_{nk}$, we have $B_n \mid B_{nk-1} B_r$.

Moreover, by Lemma 2.1 we have $\gcd(B_{nk}, B_{nk-1}) = 1$. It follows that $\gcd(B_n, B_{nk-1}) = 1$. Therefore, $B_n \mid B_r$. But this is impossible, unless $r = 0$. So, $m = nk$ and hence $n \mid m$, as desired. □

From Theorem 2.4, we obtain the following result.

Corollary 2.3. $B_n \mid B_m$ *if and only if* $n \mid m$.

Now, we have the following general result.

Theorem 2.5. *Let* m, n *be positive integers. Then,*

$$\gcd(B_n, B_m) = B_{\gcd(n,m)}.$$

To prove this theorem, we need the two following lemmas.

Lemma 2.2. *Let* m, q *be positive integers. Then* $\gcd(B_{qm-1}, B_m) = 1$.

Proof. This result follows directly from the proof of Theorem 2.4 (ii). □

Lemma 2.3. *Let* $n = mq + r, 0 \leq r < m$. *Then* $\gcd(B_n, B_m) = \gcd(B_m, B_r)$.

Proof. According to Theorem 2.1 and Lemma 2.2, we have

$$\begin{aligned}
\gcd(B_n, B_m) &= \gcd(B_{mq+r}, B_m) \\
&= \gcd(B_{qm}B_{r+1} - B_r B_{qm-1}, B_m) \\
&= \gcd(B_r B_{qm-1}, B_m) \\
&= \gcd(B_r, B_m) \\
&= \gcd(B_m, B_r).
\end{aligned}$$

□

Proof of Theorem 2.5. Without loss of generality, we can assume that $n \geq m$. Then, by the euclidean algorithm, we get the following sequence of equations

$$\begin{aligned}
n &= q_0 m + r_1, && 0 \leq r_1 < m \\
m &= q_1 r_1 + r_2, && 0 \leq r_2 < r_1 \\
r_1 &= q_2 r_2 + r_3, && 0 \leq r_3 < r_2 \\
&\vdots \\
r_{k-2} &= q_{k-1} r_{k-1} + r_k, && 0 \leq r_k < r_{k-1} \\
r_{k-1} &= q_k r_k + 0.
\end{aligned}$$

It follows from a repeated application of Lemma 2.3 that

$$\begin{aligned}
\gcd(B_n, B_m) = \gcd(B_m, B_{r_1}) &= \gcd(B_{r_1}, B_{r_2}) = \cdots \\
&= \gcd(B_{r_{k-1}}, B_{r_k}) = \gcd(B_{q_k r_k}, B_{r_k}).
\end{aligned}$$

By Theorem 2.4, $B_{r_k} \mid B_{q_k r_k}$, then $\gcd(B_{q_k r_k}, B_{r_k}) = B_{r_k}$. But $r_k = \gcd(n, m)$, so $\gcd(B_n, B_m) = B_{\gcd(n,m)}$, as desired. □

This theorem gives a quick and efficient algorithm for computing the greatest common divisor of any two balancing numbers.

Corollary 2.4. $\gcd(B_m, B_n) = 1$ *if and only if* $\gcd(m, n) = 1$.

Proof. By Theorem 2.5 and $B_1 = 1$, we have the desired proof. □

Remark 2.1. *It follows from Theorem 2.5 that the least common multiple (lcm) of B_m and B_n can also be computed quickly*

$$\text{lcm}(B_n, B_m) = \frac{B_n B_m}{\gcd(B_n, B_m)} = \frac{B_n B_m}{B_{\gcd(n,m)}}.$$

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