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OPTIMAL ATTENUATION CONTROL OF 2-D POSITIVE ROESSER SYSTEMS WITH BOUNDED DISTURBANCES

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Abstract. In this paper, performance analysis and controller design problems subject to an optimal attenuation level are studied for 2-D positive systems with bounded input disturbances. First, some novel comparison techniques for state estimations subject to peak values of external disturbances are presented to derive a characterization for l_∞-induced norm of the input-output operator. Then, we derive the necessary and sufficient linear programming (LP) conditions for obtaining a controller gain of an l_∞-induced performance with a prescribed attenuation level. Numerical examples are given to illustrate the effectiveness of the proposed method.
Keywords: 2-D systems, Roesser model, positive systems, l_∞-induced, linear programming.

1. Introduction

Positive systems theory has been extensively studied in the past few decades due to their elegant properties that have yet no counterpart in general dynamical systems [1]-[2]. Practical applications of positive systems have also been found in various areas such as economics, biology, ecology, epidemiology, and chemistry, pharmacokinetics, population dynamics or communication [3]-[5]. This theory and many problems in control theory have also been developed for some 2-D systems recently [6]-[8].

On the other hand, exogenous disturbances are unavoidable in modeling engineering systems due to many technical issues encountered in the data processing, operation and information transmission. To evaluate the effectiveness of noises, performance indicators such as l_1, l_2, H_∞, or l_∞ are widely used as the most important tools [9]-[10]. Moreover, as discussed in the existing literature, various engineering systems as wind shear on aircraft wings or continuous road excitation in vehicle suspension systems are involved external disturbances, which are only persistent and amplitude-bounded rather than specifications on the total energy of a disturbance are required [11]. Thus, the l_∞-gain minimization is a more useful and effective approach to examining the responses of dynamic systems corrupted by persistently bounded disturbances [12].

For positive systems, the use of linear co-positive Lyapunov functions [13]-[14] is one of the most popular approaches, which stimulates l_∞ performance index, to deal with the attenuation levels of persistent peak-bounded disturbances. However, considerably less attention has been devoted to formulating l_∞-gain characterization for 2-D positive systems. In addition, due to technical challenges, one cannot directly extend such studies to 2-D processes using conventional 1-D systems theory.

In this paper, we first formulate a characterization for l_∞-gain performance for 2-D positive Roesser systems. We then utilize the obtained analysis result to derive necessary and sufficient LP-based conditions for the existence of a static output-feedback l_∞-gain controller.

2. Preliminaries

Notations. $\mathbb{R}^n$ and $\mathbb{R}^{m\times n}$ denote the n-dimensional vector space and the set of $m \times n$-matrices, respectively, $1_n \in \mathbb{R}^n$ denotes the vector with all entries equal one. Max-norm of a matrix $A = (a_{ij}) \in \mathbb{R}^{m\times n}$ is defined as $\|A\|_\infty = \max_{1\le i\le m} \sum_{j=1}^{n} |a_{ij}|$. Comparisons between vectors $x, y \in \mathbb{R}^n$ are defined componentwise, that is, $x \preceq y$ if $x_i \le y_i$ and $x \prec y$ if $x_i < y_i$ for all $i \in [n]$. A matrix $A = (a_{ij}) \in \mathbb{R}^{m\times n}$ is nonnegative, written as $A \succeq 0$, if $a_{ij} \ge 0$ for all i, j and A is positive, $A \succ 0$, if $a_{ij} > 0$ for all i, j. The absolute matrix of $A = (a_{ij}) \in \mathbb{R}^{m\times n}$ is defined as $|A| = (|a_{ij}|) \in \mathbb{R}_+^{m\times n}$. l_∞-norm of a two-variable function $f : \mathbb{Z}_+ \times \mathbb{Z}_+ \to \mathbb{R}^n$ is defined as $\|f\|_{l_\infty} = \sup_{k,l\in\mathbb{Z}_+} \|f(k, l)\|_\infty$ and $l_\infty(\mathbb{R}^n) = \{f : \mathbb{Z}_+ \times \mathbb{Z}_+ \to \mathbb{R}^n : \|f\|_{l_\infty} < \infty\}$.

Consider the following 2-D system described by the Roesser model

$$\begin{bmatrix} x^h(i+1,j) \\ x^v(i,j+1) \end{bmatrix} = A \begin{bmatrix} x^h(i,j) \\ x^v(i,j) \end{bmatrix} + Bw(i,j), \tag{2.1}$$

$$z(i,j) = C \begin{bmatrix} x^h(i,j) \\ x^v(i,j) \end{bmatrix} + Dw(i,j), \tag{2.2}$$

where $x^h(i,j) \in \mathbb{R}^{n_h}$ and $x^v(i,j) \in \mathbb{R}^{n_v}$ are the horizontal and vertical state vectors, $z(i,j) \in \mathbb{R}^{n_z}$ and $w(i,j) \in \mathbb{R}^{n_w}$ are the regulated output and exogenous disturbance input vectors, respectively, A, B, C, and D are given real matrices. Initial conditions of system (2.1) are specified as

$$x^h(0,j) = \phi_h(j),\ j \in \mathbb{N}_0, \quad x^v(i,0) = \phi_v(i),\ i \in \mathbb{N}_0, \tag{2.3}$$

where the initial functions ϕ_h, ϕ_v in are assumed to have finite support, that is, there exist positive integers T_h, T_v such that $\phi_h(j) = 0$ for $j \ge T_h$ and $\phi_v(i) = 0$ for $i \ge T_v$. This assumption is widely adopted in analysis and control of 2-D systems.

Definition 2.1. *System* (2.1)-(2.2) *is said to be positive if for any initial functions ϕ_h, ϕ_v and input w, it holds that*

$$\begin{cases} \phi_h(j) \succeq 0,\ \phi_v(i) \succeq 0, \\ w(i,j) \succeq 0 \end{cases} \Longrightarrow x(i,j) \succeq 0 \text{ and } z(i,j) \succeq 0$$

for all $i, j \in \mathbb{N}_0$, where $x(i,j) = \begin{bmatrix} x^h(i,j) \\ x^v(i,j) \end{bmatrix}$ is the overall state vector.

Similar to [7], it can be shown that system (2.1)-(2.2) is positive if and only if the augmented matrix $\mathcal{M} = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$ is nonnegative.

Definition 2.2. *System* (2.1) *without exogenous disturbance (i.e. $w = 0$) is said to be globally exponentially stable (GES) if there exist scalars $\alpha \in (0,1)$ and $\beta > 0$ such that any solution $x(i,j)$ of system* (2.1) *with initial condition* (2.3) *satisfies*

$$\|x(i,j)\|_\infty \leq \beta \bigg(\sum_{k=0}^{i} \frac{\phi_v(k)}{\alpha^{k+1}} + \sum_{l=0}^{j} \frac{\phi_h(l)}{\alpha^{l+1}} \bigg) \alpha^{i+j}, \quad i, j \in \mathbb{N}_0. \tag{2.4}$$

It was derived in [7] that positive 2-D system (2.1) with $w = 0$ is GES if and only if there exists a vector $0 \prec \chi \in \mathbb{R}^n$ such that

$$A\chi - \chi \prec 0. \tag{2.5}$$

This condition is satisfied if and only if the matrix A is Schur stable, that is, the spectral radius of the matrix A satisfies $\rho(A) < 1$ (see, [15]).

We denote by $l_\infty(\mathbb{R}^{n_w})$ the space of all bounded sequences w, that is,

$$l_\infty(\mathbb{R}^{n_w}) = \left\{ w : \mathbb{Z}_+ \times \mathbb{Z}_+ \to \mathbb{R}^{n_w} \big| \|w\|_{l_\infty} < \infty \right\}.$$

In this paper, the disturbance $w(i,j)$ is assumed to belong to $l_\infty(\mathbb{R}^{n_w})$. Assume that system (2.1) without disturbance is GES. We define an input-output operator for system (2.1) as

$$T_{w,z} : l_\infty(\mathbb{R}^{n_w}) \longrightarrow l_\infty(\mathbb{R}^{n_z}), \quad w \mapsto z$$

and l_∞-gain of system (2.1) under zero initial condition is defined by

$$\|T_{w,z}\|_{l_\infty - l_\infty} = \sup_{\|w\|_{l_\infty} \neq 0} \frac{\|z\|_{l_\infty}}{\|w\|_{l_\infty}} = \sup_{\|w\|_{l_\infty} = 1} \|z\|_{l_\infty}. \tag{2.6}$$

Definition 2.3. *For a given scalar $\gamma > 0$, system* (2.1) *is said to have l_∞-gain performance of level γ if, under zero initial condition, it holds that $\|T_{w,z}\|_{l_\infty - l_\infty} < \gamma$. In other words, for any nonzero disturbance $w \in l_\infty(\mathbb{R}^{n_w})$, and under zero initial condition, the output trajectory $z(i,j)$ of system* (2.1) *satisfies*

$$\|z(i,j)\|_\infty < \gamma \|w(i,j)\|_\infty, \quad i, j \geq 0.$$

The main objective of this paper is to formulate the value of l_∞-gain $\|T_{w,z}\|_{l_\infty - l_\infty}$ for system (2.1) and characterize the l_∞-gain performance index by establishing LP-based necessary and sufficient conditions.

3. Main results

Let $x_w(i,j)$ denote the solution of (2.1) with zero initial condition corresponding to input w. For any w_1, $w_2 \in l_\infty(\mathbb{R}^{n_w})$, if $w_1 \preceq w_2$ then $x_{\tilde{w}}(i,j) = x_{w_2}(i,j) - x_{w_1}(i,j)$ is a solution of system (2.1) with zero initial condition and nonnegative input

$$\tilde{w}(i,j) = w_2(i,j) - w_1(i,j) \succeq 0.$$

Thus, due to the positivity of system (2.1), $x_{\tilde{w}}(i,j) \succeq 0$ for all $i,j \geq 0$. Based on this observation, we have the following technical lemma.

Lemma 3.1. *For any* w_1, $w_2 \in l_\infty(\mathbb{R}^{n_w})$*, if* $w_1 \preceq w_2$ *then* $x_{w_1} \preceq x_{w_2}$.

For any $w \in l_\infty(\mathbb{R}^{n_w})$, it is obvious that $-|w| \preceq w \preceq |w|$. Thus, by Lemma 3.1, we have $-x_{|w|} \preceq x_w \preceq x_{|w|}$. This shows that $|x_w(i,j)| \preceq x_{|w|}(i,j)$. In addition, for a $w \in l_\infty(\mathbb{R}^{n_w})$, we define an upper bound vector $\overline{w} = \|w\|_{l_\infty} 1_{n_w}$. Then,

$$|w(i,j)| \preceq \|w(i,j)\|_\infty 1_{n_w} \preceq \overline{w}.$$

Now, to formulate an upper estimate for $x(i,j)$ and $z(i,j)$ with respect to an input w, we consider the following positive 2-D system

$$\begin{bmatrix} \bar{x}^h(i+1,j) \\ \bar{x}^v(i,j+1) \end{bmatrix} = A \begin{bmatrix} \bar{x}^h(i,j) \\ \bar{x}^v(i,j) \end{bmatrix} + B\overline{w}, \tag{3.1}$$

$$\bar{z}(i,j) = C \begin{bmatrix} \bar{x}^h(i,j) \\ \bar{x}^v(i,j) \end{bmatrix} + D\overline{w}, \tag{3.2}$$

where $w \in l_\infty(\mathbb{R}^{n_w})$ and $\overline{w} = \|w\|_{l_\infty} 1_{n_w}$.

Lemma 3.2. *For any* $w \in l_\infty(\mathbb{R}^{n_w})$*,* $w \succeq 0$*, let* x_w*,* $\bar{x}_w$ *and* z_w*,* $\bar{z}_w$ *be the state and output trajectories of systems* (2.1)-(2.2) *and* (3.1)-(3.2) *with zero initial condition. The, we have*

$$0 \preceq x_w \preceq \bar{x}_w \text{ and } 0 \preceq z_w \preceq \bar{z}_w.$$

Proof. Let $\bar{e}(i,j) = \bar{x}_w(i,j) - x_w(i,j)$. It follows from (2.1) and (3.1) that

$$\begin{bmatrix} \bar{e}^h(i+1,j) \\ \bar{e}^v(i,j+1) \end{bmatrix} = A \begin{bmatrix} \bar{e}^h(i,j) \\ \bar{e}^v(i,j) \end{bmatrix} + B\left(\overline{w} - w(i,j)\right). \tag{3.3}$$

Since $\overline{w} - w(i,j) \succeq 0$ and system (3.3) is positive, we have $\bar{e}(i,j) \succeq 0$ and, thus, $x_w(i,j) \preceq \bar{x}_w(i,j)$ for all $i,j \geq 0$. In addition, it follows from (2.2) and (3.2) that

$$\bar{z}_w(i,j) - z_w(i,j) = C\bar{e}(i,j) + D\left(\overline{w} - w(i,j)\right) \succeq 0,$$

which ensures that $z_w(i,j) \preceq \bar{z}_w(i,j)$. □

To establish a lower bound for $x(i,j)$ and $z(i,j)$, similar to (3.1)-(3.2), we consider the following system

$$\begin{bmatrix}\underline{x}^h(i+1,j)\\ \underline{x}^v(i,j+1)\end{bmatrix} = A\begin{bmatrix}\underline{x}^h(i,j)\\ \underline{x}^v(i,j)\end{bmatrix} + B\underline{w}, \tag{3.4}$$

$$\underline{z}(i,j) = C\begin{bmatrix}\underline{x}^h(i,j)\\ \underline{x}^v(i,j)\end{bmatrix} + D\underline{w}, \tag{3.5}$$

where $\underline{w} \in \mathbb{R}_+^{n_w}$ is a constant vector such that $w(i,j) \succeq \underline{w}$.

Lemma 3.3. *For a $w \in l_\infty(\mathbb{R}^{n_w})$ and a vector $\underline{w} \in \mathbb{R}_+^{n_w}$ such that $w(i,j) \succeq \underline{w}$, let x_w, $\underline{x}_w$ and z_w, $\underline{z}_w$ be the state and output trajectories of systems* (2.1)-(2.2) *and* (3.4)-(3.5) *with zero initial condition. Then, we have*

$$0 \preceq \underline{x}_w \preceq x_w \text{ and } 0 \preceq \underline{z}_w \preceq z_w.$$

Assume that the matrix A of (2.1) is Schur stable. Then, the matrix $I_n - A$ is invertible and we have

$$(I_n - A)^{-1} = \sum_{k=0}^{\infty} A^k \succeq 0.$$

A constant vector $x_e \in \mathbb{R}_+^n$ is said to be a positive equilibrium (PE) of system (3.1) if it satisfies the following algebraic equation

$$x_e = Ax_e + B\overline{w}. \tag{3.6}$$

It can be verified from (3.1) and (3.6) that, subject to condition $\rho(A) < 1$, the unique PE of system (3.1) can be represented as $x_e = (I_n - A)^{-1} B\overline{w}$.

Lemma 3.4. *For any $w \in l_\infty(\mathbb{R}^{n_w})$, the solution x_w of* (2.1) *with zero initial condition satisfies*

$$|x_w(i,j)| \preceq \|w\|_{l_\infty} (I_n - A)^{-1} B 1_{n_w} = x_e. \tag{3.7}$$

In particular, x_w also belongs to $l_\infty(\mathbb{R}^n)$.

Proof. Since $|w(i,j)| \preceq \overline{w}$, by Lemmas 3.1 and 3.2, we have

$$|x_w(i,j)| \preceq x_{|w|}(i,j) \preceq \bar{x}_w(i,j). \tag{3.8}$$

Let $\hat{e}(i,j) = x_e - \bar{x}_w(i,j)$. Then, it follows from (3.1) and (3.7) that

$$\begin{bmatrix}\hat{e}^h(i+1,j)\\ \hat{e}^v(i,j+1)\end{bmatrix} = A\begin{bmatrix}\hat{e}^h(i,j)\\ \hat{e}^v(i,j)\end{bmatrix}. \tag{3.9}$$

System (3.9) is positive and $\hat{e}(0,0) = x_e \succeq 0$. Thus, $\hat{e}(i,j) \succeq 0$. This, together with (3.8), gives $|x_w(i,j)| \preceq x_e$, by which we can obtain

$$\|x_w\|_{l_\infty} = \sup_{i,j\geq 0} \|x_w(i,j)\|_\infty \leq \|x_e\|_\infty.$$

The proof is completed. □

Remark 3.1. *It can be shown under condition* (2.5) *that the positive 2-D system* (3.9) *is GES. Thus, the state trajectory* $\bar{x}_w(i,j)$ *of* (3.1) *increases exponentially to* x_e. *More precisely, there exists an* $\alpha \in (0,1)$ *such that*

$$0 \preceq x_e - \bar{x}_w(i,j) = \hat{e}(i,j) \preceq x_e \alpha^{i+j} \tag{3.10}$$

for a given $w \in l_\infty(\mathbb{R}^{n_w})$. *From* (3.10)*, we obtain the interval estimate*

$$\left(1 - \alpha^{i+j}\right) x_e \preceq \bar{x}_w(i,j) \preceq x_e, \tag{3.11}$$

where $x_e = \|w\|_{l_\infty} \left(I_n - A\right)^{-1} B 1_{n_w}$.

Theorem 3.1. *Assume that the 2-D system* (2.1) *is positive and GES. The value of* l_∞*-gain of system* (2.1) *under zero initial condition can be expressed as*

$$\|T_{w,z}\|_{l_\infty - l_\infty} = \left\|C\left(I_n - A\right)^{-1} B + D\right\|_\infty. \tag{3.12}$$

Proof. For any $w \in l_\infty(\mathbb{R}^{n_w})$, by Lemma 3.2, we have

$$|z_w(i,j)| \preceq z_{|w|}(i,j) \preceq \bar{z}(i,j).$$

We will show that

$$\bar{z}(i,j) \preceq \|w\|_{l_\infty} \left[C\left(I_n - A\right)^{-1} B + D\right] 1_{n_w}.$$

Indeed, as shown in the proof of Lemma 3.4 that $\bar{x}(i,j) \preceq x_e$ for all $i, j \geq 0$. This, together with (3.2), gives

$$\bar{z}(i,j) \preceq C x_e + D\overline{w} = \Psi 1_{n_w} \|w\|_{l_\infty}, \tag{3.13}$$

where $\Psi = C\left(I_n - A\right)^{-1} B + D$. From (3.13) and the fact $z_w(i,j) \preceq \bar{z}(i,j)$, we obtain

$$\|z_w(i,j)\|_\infty \leq \|\bar{z}(i,j)\|_\infty \leq \|\Psi 1_{n_w}\|_\infty \|w\|_{l_\infty}. \tag{3.14}$$

It follows from (3.14) that

$$\|T_{w,z}\|_{l_\infty - l_\infty} = \sup_{w \neq 0} \frac{\|z_w\|_{l_\infty}}{\|w\|_{l_\infty}} \leq \|\Psi \mathbf{1}_{n_w}\|_\infty = \|\Psi\|_\infty. \tag{3.15}$$

To complete the proof, we will show that $\|T_{w,z}\|_{l_\infty - l_\infty} \geq \|\Psi\|_\infty$. Let $w_0 \in l_\infty(\mathbb{R}^{n_w})$ be a given sequence with $\|w_0\|_{l_\infty} > 0$. We define

$$w(i,j) = \underline{w} = \|w_0\|_{l_\infty} 1_{n_w},\ i, j \geq 0.$$

By Lemma 3.3, we have

$$z_w(i,j) \succeq \underline{z}_w(i,j),\ i, j \geq 0.$$

In addition, with the input $\underline{w}$, the unique PE of (3.4) is given by

$$\underline{x}_e = (I_n - A)^{-1} B\underline{w}.$$

Similar to the proof of Lemma 3.4 and Remark 3.1, we can conclude under condition (2.5) that there exists a scalar $\hat{\alpha} \in (0, 1)$ such that

$$\left(1 - \hat{\alpha}^{i+j}\right) \underline{x}_e \preceq \underline{x}_w(i, j) \preceq \underline{x}_e. \tag{3.16}$$

Therefore, from (3.5) and (3.16), we obtain

$$\underline{z}_w(i, j) \succeq \hat{\theta}_{i,j} C\underline{x}_e + D_w\underline{w}, \tag{3.17}$$

where $\hat{\theta}_{i,j} = 1 - \hat{\alpha}^{i+j}$. From (3.17), we get

$$\begin{aligned} \sup_{i,j\geq 0} \|\underline{z}_w(i, j)\|_\infty &\geq \limsup_{i+j\to\infty} \left\|\hat{\theta}_{i,j} C\underline{x}_e + D\underline{w}\right\|_\infty \\ &= \|C\underline{x}_e + D\underline{w}\|_\infty \\ &= \|\Psi 1_{n_w}\|_\infty \|w_0\|_{l_\infty} \\ &= \|\Psi\|_\infty \|w_0\|_{l_\infty}. \end{aligned} \tag{3.18}$$

It follows from (3.18) and the fact $\|z_w\|_{l_\infty} \geq \sup_{i,j\geq 0} \|\underline{z}_w(i, j)\|_\infty$ that

$$\|T_{w,z}\|_{l_\infty - l_\infty} = \sup_{\|w_0\|_{l_\infty}=1} \|z_w\|_{l_\infty} \geq \|\Psi\|_\infty. \tag{3.19}$$

The results of (3.15) and (3.19) ensure that $\|T_{w,z}\|_{l_\infty - l_\infty} = \|\Psi\|_\infty$. The proof is completed. □

Based on Theorem 3.1, we now derive tractable necessary and sufficient conditions for the design problem of an l_∞-gain controller. More precisely, we formulate LP-based conditions such that, for a given attenuation level $\gamma > 0$, the closed-loop l_∞-gain satisfies $\|T_{w,z}\|_{l_\infty - l_\infty} < \gamma$.

Theorem 3.2. *For a given $\gamma > 0$, the positive 2-D system* (2.1) *is GES and has l_∞-gain performance at level γ if and only if there exists a positive vector $\chi \in \mathbb{R}^n$ that satisfies the following LP-based conditions*

$$(A - I_n)\chi + B1_{n_w} \prec 0, \tag{3.20}$$

$$C\chi + D1_{n_w} - \gamma 1_{n_z} \prec 0. \tag{3.21}$$

Proof. (Necessity) Let $\chi_0 \in \mathbb{R}^n$ be a positive vector that satisfies the stability condition (2.5). For sufficiently small $\epsilon > 0$, we define

$$\chi = \epsilon\chi_0 + (I_n - A)^{-1} B1_{n_w}.$$

It is clear that $\chi \succ 0$ and we have

$$(A - I_n)\chi + B1_{n_w} = \epsilon(A - I_n)\chi_0 \prec 0. \tag{3.22}$$

On the other hand, by Theorem 3.1, $\|T_{w,z}\|_{l_\infty - l_\infty} < \gamma$ if and only if

$$\Gamma = \left[C(I_n - A)^{-1}B + D\right] 1_{n_w} \prec \gamma 1_{n_z}.$$

Thus, $\zeta = \gamma 1_{n_w} - \Gamma \succ 0$ and

$$C\chi + D1_{n_w} - \gamma 1_{n_z} = \epsilon C\chi_0 - \zeta \prec 0. \tag{3.23}$$

(Sufficiency) Since $B1_{n_w} \succeq 0$, condition (3.20) implies (2.5). Thus, system (2.1) with $w = 0$ is GES. Let

$$\tilde{\chi} = (A - I_n)\,\chi + B1_{n_w}$$

then $\tilde{\chi} \prec 0$ and $\chi = (A - I_n)^{-1}\left(\tilde{\chi} - B1_{n_w}\right)$. It follows from (3.21) that

$$\begin{aligned} \gamma 1_{n_z} &\succ C\chi + D1_{n_w} \\ &= C(A - I_n)^{-1}\tilde{\chi} + \Psi 1_{n_w}, \end{aligned} \tag{3.24}$$

where the matrix Ψ is defined in the proof of Theorem 3.1.

For any $w \in l_\infty(\mathbb{R}^{n_w})$ with $\|w\|_{l_\infty} = 1$, since $(A - I_n)^{-1}\tilde{\chi} \succ 0$, from (3.24), we obtain

$$\|z\|_{l_\infty} \leq \|\Psi 1_{n_w}\|_\infty < \gamma.$$

The proof is completed. □

The following alternative performance conditions can be obtained by similar arguments used in the proof of Theorem 3.2.

Theorem 3.3. *For a given $\gamma > 0$, the positive system (2.1) is GES and has l_∞-gain performance at level γ if and only if the following LP-based conditions are feasible for a vector $0 \prec \eta \in \mathbb{R}^n$*

$$\eta^\top (A - I_n) + 1_{n_z}^\top C \prec 0, \tag{3.25}$$

$$\eta^\top B + 1_{n_z}^\top D - \gamma 1_{n_w}^\top \prec 0. \tag{3.26}$$

4. Static-output feedback l_∞-gain control

Consider the following control system

$$\begin{aligned} \begin{bmatrix} x^h(i+1,j) \\ x^v(i,j+1) \end{bmatrix} &= A \begin{bmatrix} x^h(i,j) \\ x^v(i,j) \end{bmatrix} + B_u u(i,j) + Bw(i,j), \\ z(i,j) &= C \begin{bmatrix} x^h(i,j) \\ x^v(i,j) \end{bmatrix} + D_u u(i,j) + Dw(i,j), \\ x_{mes}(i,j) &= Mx(i,j) + Nw(i,j), \end{aligned} \tag{4.1}$$

where $u(i,j) \in \mathbb{R}^{n_u}$ is the control input and $x_{mes}(i,j) \in \mathbb{R}^{n_o}$ is the measurement output vector, B_u, D_u, M, N are given real matrices.

A static output-feedback controller (SOFC) will be designed in the form

$$u(i,j) = -Kx_{mes}(i,j) \tag{4.2}$$

to make the closed-loop system positive, GES and admit a prescribed l_∞-gain performance, where $K \in \mathbb{R}^{n_u \times n_o}$ is the controller gain. The closed-loop system of (4.1) subject to SOFC (4.2) is obtained as

$$\begin{aligned} \begin{bmatrix} x^h(i+1,j) \\ x^v(i,j+1) \end{bmatrix} &= A_c \begin{bmatrix} x^h(i,j) \\ x^v(i,j) \end{bmatrix} + B_c w(i,j), \\ z(i,j) &= C_c \begin{bmatrix} x^h(i,j) \\ x^v(i,j) \end{bmatrix} + D_c w(i,j), \end{aligned} \tag{4.3}$$

where $A_c = A - B_uKM$, $B_c = B - B_uKN$, $C_c = C - D_uKM$ and $D_w = D - D_uKN$.

First, it can be verified that system (4.3) is positive if and only if

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} - \begin{bmatrix} B_u \\ D_u \end{bmatrix} K \begin{bmatrix} M & N \end{bmatrix} \succeq 0. \tag{4.4}$$

In addition, for a given $\gamma > 0$, by Theorem 3.2, system (4.3) is GES and has l_∞-gain performance at level γ if and only if the LP-based condition

$$\left(\mathcal{A} - \begin{bmatrix} B_u \\ D_u \end{bmatrix} K \begin{bmatrix} M & N \end{bmatrix} \right)^\top \begin{bmatrix} \chi \\ \mathbf{1}_{n_z} \end{bmatrix} \prec \gamma \begin{bmatrix} 0 \\ \mathbf{1}_{n_w} \end{bmatrix} \tag{4.5}$$

is feasible for a vector $0 \prec \chi \in \mathbb{R}^n$, where $\mathcal{A} = \begin{bmatrix} A - I_n & B \\ C & D \end{bmatrix}$.

In the following, based on the ideas of vertex optimization proposed in [7], we derive optimal conditions for the existence of a controller gain K satisfying (4.4)-(4.5) for the case of single-input systems (i.e. $n_u = 1$).

Let $G = \begin{bmatrix} B \\ D \end{bmatrix} = (g_i) \in \mathbb{R}_+^{n+n_o}$ and assume $G \neq 0$ (i.e. at least one component $g_i > 0$). We denote

$$\mathcal{M} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} = (m_{ij}) \in \mathbb{R}_+^{(n+n_z)\times(n+n_w)}$$

and decompose the matrix

$$H = \begin{bmatrix} M & N \end{bmatrix} = \begin{bmatrix} h_1 & h_2 & \cdots & h_{n+n_w} \end{bmatrix},$$

where $h_j \in \mathbb{R}^{n_o}$.

The controller gain matrix can be specified as $K = k^\top$, $k \in \mathbb{R}^{n_o}$. Thus, condition (4.5) holds if and only if

$$k^\top h_j \leq \theta_j^* \triangleq \min_{1\leq i\leq n+n_z} \left\{ \frac{m_{ij}}{g_i} : g_i > 0 \right\}. \tag{4.6}$$

According to (4.6), we define the polyhedron Δ_P by

$$\Delta_P = \left\{ k \in \mathbb{R}^{n_o} \middle| k^\top H \preceq \theta^* \right\}, \tag{4.7}$$

where $\theta^* = \begin{bmatrix} \theta_1^* & \theta_2^* & \cdots & \theta_{n+n_w}^* \end{bmatrix}$. Assume that the matrix $H = \begin{bmatrix} M & N \end{bmatrix}$ has full-row rank. Then, equation (4.7) defines a nonempty convex polyhedron. Let $\mathcal{V}$ be the set of vertices of Δ_P.

For a fixed vector $\chi \succ 0$, the function $\varphi_\chi(k) = k^\top H\chi$ is continuous and, thus, attains its maximum on the compact $\Omega = \Delta_P \cap \{k \in \mathbb{R}^{n_o} : \varphi_\chi(k) \geq 0\}$.

Lemma 4.1. *There exists a vertex $k_v \in \mathcal{V}$ such that $\varphi_\chi(k_v) = \max_{k\in\Delta_P} \varphi_\chi(k)$.*

Proof. See, for example, Lemma 3 in [7]. □

The following theorem gives a criterion in terms of LP-based conditions for the existence of a desired l_∞-gain controller (4.2).

Theorem 4.1. *For a given $\gamma > 0$, there exists an SOFC in the form of* (4.2) *that makes the closed-loop system* (4.3) *positive, GES and admit an l_∞-gain performance at level γ if and only if there exists a vertex $k_v^* \in \mathcal{V}$ of Δ_P such that the LP-based problem*

$$\left(\begin{bmatrix} A - I_n & B \\ C & D \end{bmatrix} - \begin{bmatrix} B_u \\ D_u \end{bmatrix} k_v^{*\top} \begin{bmatrix} M & N \end{bmatrix} \right) \begin{bmatrix} \chi \\ 1_{n_w} \end{bmatrix} \prec \gamma \begin{bmatrix} 0 \\ 1_{n_z} \end{bmatrix} \tag{4.8}$$

is feasible for a positive vector $\chi \in \mathbb{R}^n$.

Proof. The *Sufficiency* is obvious. We now prove the *Necessity*. Let $K = k_0^\top$ be a desired controller gain. Then, by (4.4)-(4.7), $k_0 \in \Delta_P$.

If $k_0 \notin \mathcal{V}$, we consider the optimization problem

$$\texttt{maximize} \quad \varphi_\chi(k) = k^\top H \begin{bmatrix} \chi \\ 1_{n_w} \end{bmatrix} \quad \texttt{s.t.} \quad k \in \Delta_P. \tag{4.9}$$

By Lemma 4.1, there exists a vertex $k_v^* \in \mathcal{V}$ such that

$$\varphi_\chi(k_v^*) = \max_{k\in\Delta_P} \varphi_\chi(k).$$

Since $\varphi_\chi(k) \leq \varphi_\chi(k_v^*)$ for all $k \in \Delta_P$, we have

$$\left(\mathcal{A} - Gk_v^{*\top} H\right) \begin{bmatrix} \eta \\ 1_{n_w} \end{bmatrix} \preceq \left(\mathcal{A} - GKH\right) \begin{bmatrix} \eta \\ 1_{n_w} \end{bmatrix}$$

for any K satisfying (4.4), where

$$\mathcal{A} = \begin{bmatrix} A - I_n & B \\ C & D \end{bmatrix} \text{ and } G = \begin{bmatrix} B_u \\ D_u \end{bmatrix}.$$

The above inequality validates condition (4.8). The proof is completed. □

Remark 4.1. *It should be clarified that the controller synthesis conditions derived in Theorem 4.1 are restricted to single-input systems (i.e. $u(i,j) \in \mathbb{R}^1$). Potential extensions to positive 2-D systems with multiple inputs can be suitably developed utilizing the method proposed in [7]. However, it requires further technical development as l_∞-gain can be regarded as a dual setting of l_1-gain. This motivates some future work.*

Remark 4.2. *For single-input single-output systems, an optimal controller gain is obtained explicitly as*

$$K = k^*_{\text{op}} = \min\left\{\frac{m_{ij}}{g_i h_j} : g_i h_j > 0\right\}$$

and a desired SOFC (4.2) *exists if and only if there exists a vector* $0 \prec \chi \in \mathbb{R}^n$ *such that*

$$\left(\mathcal{A} - k^*_{op} GH\right) \begin{bmatrix} \chi \\ 1_{n_w} \end{bmatrix} \prec \gamma \begin{bmatrix} 0 \\ 1_{n_z} \end{bmatrix}. \tag{4.10}$$

5. Illustrative examples

Example 5.1. *Consider a 2-D system as given in* (2.1)-(2.2) *with the matrices*

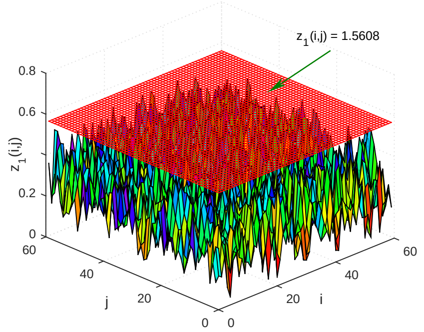
$$A = \begin{bmatrix} 0.15 & 0.03 & 0.05 \\ 0.18 & 0.25 & 0.35 \\ 0.2 & 0.15 & 0.2 \end{bmatrix}, \quad B = \begin{bmatrix} 0.25 \\ 0.15 \\ 0.05 \end{bmatrix},$$
$$C = \begin{bmatrix} 0.25 & 0.35 & 0.48 \\ 0.5 & 0.6 & 0.15 \end{bmatrix}, \quad D = \begin{bmatrix} 0.35 \\ 0.16 \end{bmatrix}.$$

By using a the MATLAB `linprog` *Toolbox, it is found that the LP-based conditions* (3.20)-(3.21) *are feasible with* $\gamma \geq \gamma_{\min} \triangleq 0.5608$.

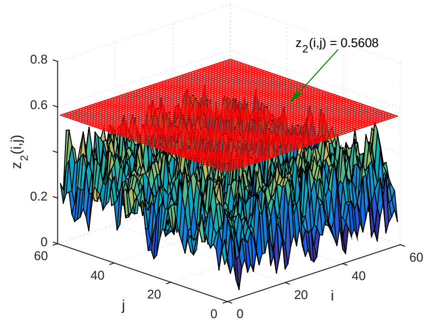
To support the illustration of the analysis results, we conduct some state trajectories of output $z(i,j)$ with respect to $\|w\|_{l_\infty} = 1$ and zero initial condition. The obtained results are presented in Figure 1 (a)-(b). In addition, the corresponding trajectory of input $w(i,j)$ is drawn in Figure 2. The simulation results in Figures 1 and 2 indicate that $\|z\|_{l_\infty} < \gamma_{\min}\|w\|_{l_\infty}$ as revealed by the obtained theoretical results.

Example 5.2. *Consider system* (4.1) *with the following system matrices*

$$A = \begin{bmatrix} 1.0 & 0.2 & 0.15 \\ 0.2 & 0.4 & 0.45 \\ 0.2 & 0.35 & 0.7 \end{bmatrix}, \quad B_u = \begin{bmatrix} 1.0 \\ 0 \\ 1.0 \end{bmatrix}, \quad B = \begin{bmatrix} 0.1 \\ 0 \\ 0.15 \end{bmatrix}$$
$$C = \begin{bmatrix} 0.2 & 0.25 & 0.3 \\ 0.4 & 0.19 & 0.25 \end{bmatrix}, \quad D = \begin{bmatrix} 0.15 \\ 0.2 \end{bmatrix}, \quad M = \begin{bmatrix} 1.0 & 1.0 & 0 \end{bmatrix}, \quad N = 0.1.$$



(a) $z_1(i,j)$



(b) $z_2(i,j)$

Figure 1. State trajectories of $z_1(i,j)$ ***and*** $z_2(i,j)$ ***with zero initial condition and*** $\|w\|_{l_\infty} = 1$

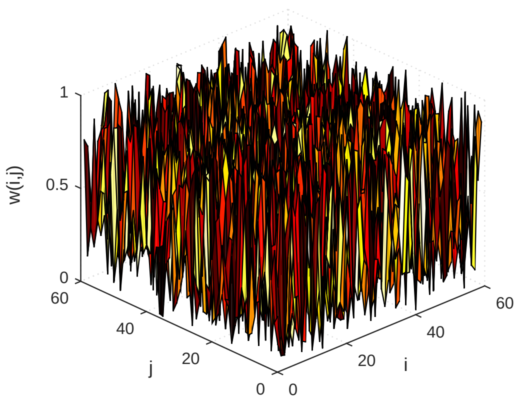


Figure 2. A state trajectory of disturbance $w(i,j)$ ***with*** $\|w\|_{l_\infty} = 1$

Note at first that the unforced system (without control) is unstable. Figure 3 presents a state trajectory $\|z(i,j)\|_\infty$ *of the output under zero initial condition. It can be seen from Figure 3 that the output goes to infinity as* $i + j \to \infty$*.*

We apply the design method of Theorem 4.1. It can be verified that the set $\mathcal{V}$ *is singleton* $\{k^*_{\text{op}}\}$*, where*

$$k^*_{\text{op}} = \min\left\{ \frac{m_{ij}}{g_j h_j} \;\middle|\; g_i h_j \neq 0 \right\} = 0.2$$

and condition (4.8) *is feasible for a vector* $\chi \succ 0$ *if and only if* $\gamma > \gamma_{\min} = 1.54$*. By Theorem 4.1, the closed-loop system* (4.3) *is positive, GES and has* l_∞*-gain performance at level* $\gamma > \gamma_{\min}$*.*

State trajectories $z_1(i,j)$ *and* $z_2(i,j)$ *of the closed-loop system with* $\|w\|_{l_\infty} = 1$ *under zero initial condition are presented in Figure 4. The simulation result in Figure 4 shows the effectiveness of the analysis results.*

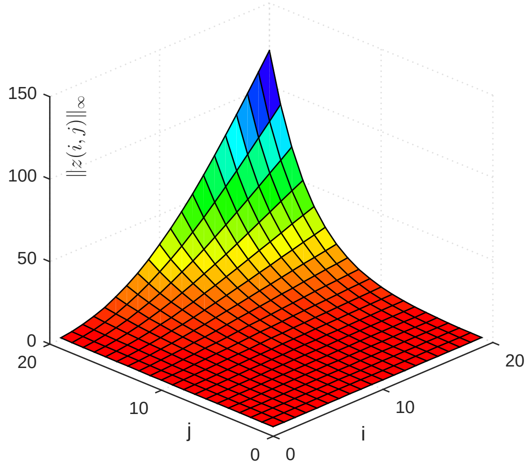
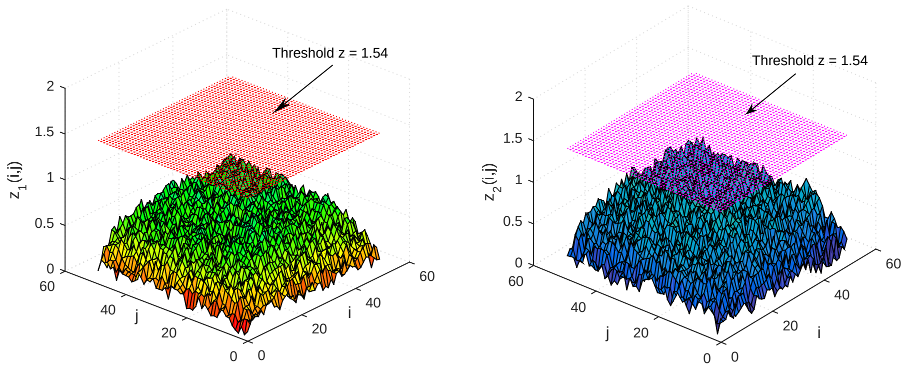


Figure 3. An open-loop trajectory $\|z(i,j)\|_\infty$ with $\|w\|_{l_\infty} = 1$ and zero initial condition



(c) $z_1(i,j)$ (d) $z_2(i,j)$

Figure 4. State trajectories of $z_1(i,j)$ and $z_2(i,j)$ of the closed-loop system with zero initial condition and $\|w\|_{l_\infty} = 1$

6. Conclusions

In this paper, the problems of performance analysis and controller design subject to optimal attenuation level have been addressed for 2-D positive systems with bounded disturbances. A characterization of l_∞-induced norm of the input-output operator has been formulated and LP-based conditions for l_∞-induced performance of the system with a prescribed attenuation level have been formulated. As an application, the problem of l_∞-gain control via static output-feedback controllers has also been discussed. A numerical example with simulations has been provided to illustrate the effectiveness of the proposed method.

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