HNUE JOURNAL OF SCIENCE
Natural Science, 2025, Volume 70, Issue 4, pp. 32-42
This paper is available online at http://hnuejs.edu.vn/ns
DOI: 10.18173/2354-1059.2025-0052

DIFFERENTIAL VARIATIONAL INEQUALITIES VIA THE EXTENDED LASALLE'S INVARIANCE PRINCIPLE APPROACH

Hoang Mai Huong[1] and Nguyen Thi Nhung[2,*]
[1]*Department of Mathematics and Information Technology, Hanoi Medical University, Hanoi city, Vietnam*
[2]*Trang An Pedagogical Practice High School, Hoa Lu University, Ninh Binh province, Vietnam*
[*]Corresponding author: Nguyen Thi Nhung, e-mail: nguyennhunghnue277@gmail.com

Received August 11, 2025. Revised October 28, 2025. Accepted December 30, 2025.

Abstract. In this work, we establish an extended LaSalle's invariance for a class of nonautonomous differential inclusions in Euclidean spaces. We also give sufficient conditions for the stability of an equilibrium of differential variational inequalities. Special cases as linear complementarity problems, based on LaSalle's invariance in both autonomous and non-autonomous cases, are also studied in this paper.

Keywords: differential variational inequality; LaSalle's invariance principle; Lyapunov functions.

1. Introduction

Differential variational inequalities (abbreviated by DVIs) have been combined by two components: differential equations/inlusions and variational inequalities. The DVI is a new modeling paradigm for many important applications in engineering and economics which presents dynamics mixed constraints in the form of variational inequalities, equilibrium conditions in a systematic way [1], [2]. These systems extend the notion of differential complementarity problems [3], dynamical Nash equilibrium problems [4], parameter estimation in metabolic flux balance models [5] and have been studied by numerous authors [6]-[9], etc.

In this paper, we are concerned with the nonlinear differential variational inequality of the following form

$$x'(t) \in Ax(t) + B(t, x(t), u(t)), t > 0, \quad (1.1)$$

$$\langle v - u(t), F(x(t)) + G(u(t)) \rangle \geq 0, \forall v \in K, \text{ for a.e. } t \geq 0, \quad (1.2)$$

$$x(0) = \xi, \tag{1.3}$$

where A is a linear operator, B, F, G are given maps. It should be highlighted that, when (1.1) is a differential equation, problem (1.1)-(1.3) is called a differential system with unilateral constraints in the theory of differential equations. Such problems can be seen as a control system subject to constraints.

An approach to tackle the above problem is to transfer (1.1)-(1.3) to a differential inclusion. By substituting u from (1.2) into the inclusion (1.1), we get

$$x'(t) \in Ax(t) + B(t, x(t), SOL(K, F(x(t)) + G(\cdot))),$$

where $SOL(K, z + G(\cdot))$ refers the solution set of the variational inequality $\langle v - u, z + G(u(t)) \rangle \geq 0, \forall v \in K$.

One of the most widely adopted stability concepts is Lyapunov stability, which plays important roles in systems and control theory and in the analysis of engineering systems (see [10]). LaSalle's invariance principle is an important extension which applies successfully in the stability analysis of autonomous smooth systems (see, e.g., [11]). In the case of nonautonomous differential equations, the author handled the stability of solutions with the help of limit differential inclusions (see [12]-[15]). However, to the best of our knowledge, there is no effort on the invariance principle of LaSalle-type for nonautonomous differential inclusions, and this fact is the main goal in the present paper.

Our contribution is to establish an extension of LaSalle's invariance principle for nonautonomous differential inclusions and to give some applications to some class of differential variational inequalities. Consider the differential inclusion of the form:

$$\dot{x}(t) \in \Phi(t, x(t)), \tag{1.4}$$

where Φ is a multivalued map with suitable conditions. A natural question is the following: If $V(x)$ is a Lyapunov function such that $\dot{V}(x) \leq 0$, is it possible to conclude that a bounded solution $x(t; x_0)$ converges to the set E defined by $\dot{V}(x) = 0$? In general, the answer is no and a direct application of LaSalle's invariance principle is not very helpful in such a situation because the limit set of a nonautonomous system is not invariant. We need to transfer our problem to an autonomous problem by the sense of limiting differential inclusions.

The remainder of this paper is organized as follows. In Section 2, we establish an extended LaSalle's invariance principle. In Section 3, we use the obtained results of Section 2 to nonautonomous differential variational inequalities and differential linear complementarity problems.

2. Extended LaSalle's invariance principle

2.1. Stability

In this section, we recall the concepts of stability and asymptotic stability for the zero solution of a differential inclusion and consider some methods that may be used to prove the stability.

Let $\Phi : J := \mathbb{R}^+ \times \mathbb{R}^n \to \mathbb{R}^n$ be given where $0 \in \Phi(t, 0)$ for a.e $t \geq 0$. We consider the following differential inclusion:

$$x'(t) \in \Phi(t, x(t)), t > 0. \tag{2.1}$$

In what follows, we use the assumptions that

(A1) Φ has nonempty, convex, compact values and $\Phi(t, \cdot)$ is u.s.c for all $t \geq 0$, $\Phi(\cdot, x)$ is measurable for each $x \in \mathbb{R}^n$.

(A2) There exists a function $\eta(\cdot) \in L^1(J, \mathbb{R}^n)$ such that $\|\Phi(t, x)\| \leq \eta(t)(1 + \|x\|)$ for all $x \in \mathbb{R}^n$, for a.e. $t \in J$.

Thus, if the assumption **(A1)** and **(A2)** are satisfied, then the inclusion (2.1) with any initial data in $\mathbb{R}^n$ has at least one solution.

Let $\mathcal{S}(\xi)$ be the set of solutions starting at ξ and $\mathcal{W}$ be the set of stationary solutions of (2.1):

$$\mathcal{N} = \{y \in \mathbb{R}^n : 0 \in \Phi(t, y), \text{ a.e. } t \geq 0\}.$$

Then we have $0 \in \mathcal{N}$.

Definition 2.1. *The equilibrium point $x = 0$ is said to be*

(i) stable if for any $\epsilon > 0$, there exists a $\delta(\epsilon) > 0$ such that for all $\xi \in \mathbb{B}_{\delta(\epsilon)}$, $\|x(t; \xi)\| \leq \epsilon, \forall t \geq 0$;

(ii) for any $\epsilon > 0$, there exists a $\delta(\epsilon) > 0$ such that for all $\xi \in \mathbb{B}_{\delta(\epsilon)}$, $\lim\limits_{t\to\infty} \|x(t; \xi)\| = 0$;

(iii) globally attractive if for all $\xi \in \mathbb{R}^n$, $\lim\limits_{t\to\infty} \|x(t; \xi)\| = 0$;

(iv) asymptotically stable if it is stable and attractive;

(v) globally asymptotically stable if it is stable and globally attractive.

Let us denote the set-valued orbital derivative of a continuously differentiable function $V : \bar{\mathbb{B}}_\sigma \in \mathbb{R}^n \to \mathbb{R}$ (for some $\sigma > 0$) with respect to the differential inclusion (2.1):

$$\dot{V}_\Phi(t, x) = \{p \in \mathbb{R} : \exists \omega \in \Phi(t, x) \text{ such that } p = \langle V'(x), \omega\rangle\}.$$

The upper and lower orbital derivatives of V with respect to the differential inclusion (2.1) are sequentially defined by

$$\dot{V}^*_\Phi(t, x) = \max_{\omega \in \Phi(t,x)} \langle V'(x), \omega\rangle, \quad \dot{V}_{*\Phi}(t, x) = \min\nolimits_{\omega \in \Phi(t,x)} \langle V'(x), \omega\rangle.$$

Remark 2.1. *(1) We have $\dot{V}_\Phi(t,x)$ is a non-empty, convex compact subset in $\mathbb{R}$. Therefore, $\dot{V}_\Phi(t,x)$ is of the following form: $\dot{V}_\Phi(t,x) = [\dot{V}_{*\Phi}(t,x), \dot{V}^*_\Phi(t,x)]$.*

(2) If $x(t)$ is a solution of (2.1) *then:*

$$\frac{d}{dt}V(x(t)) \in \dot{V}_\Phi(t,x(t)), \text{ a.e. } t \geq 0.$$

(3) Let $y \in \mathcal{N}$, i.e. $0 \in \Phi(t,y)$ a.e. $t \geq 0$. By definition of $\dot{V}_\Phi$, it is easy to see that $0 \in \dot{V}_\Phi(t,y)$ for a.e. $t \geq 0$. It means that $\mathcal{N} \subset \mathcal{Z} := \{y \in \mathbb{R}^n : 0 \in \dot{V}_\Phi(t,y), \text{ a.e. } t \geq 0\}$.

(4) In the autonomous case, i.e. Φ does not depend on t, the set-valued orbital derivative and the upper and lower orbital derivatives of V also do not depend on t, and we also receive the results as a special case of nonautonomous differential inclusions.

Definition 2.2. *(i) Let $V : \bar{\mathbb{B}}_\sigma \in \mathbb{R}^n \to \mathbb{R}$ be a continuous function such that $V(0) = 0$. We say that V is positive definite if $V(0) > 0$ for all $x \in \bar{\mathbb{B}}_\sigma \setminus \{0\}$.*

(ii) A Lyapunov function for (2.1) *is a positive definite continuously differentiable function $V : \bar{\mathbb{B}}_\sigma \in \mathbb{R}^n \to \mathbb{R}$ such that $\dot{V}^*_\Phi(t,x) \leq 0$ for all $x \in \bar{\mathbb{B}}_\sigma \setminus \{0\}$, for a.e. $t \geq 0$.*

Theorem 2.1. *Suppose that (A1) and (A2) hold. If there is a Lyapunov function V associated with the problem* (2.1)*, then the trivial solution is asymptotically stable.*

Proof. Since $V : \bar{\mathbb{B}}_\sigma \to \mathbb{R}$ is a Lyapunov function for (2.1), V is positive definite continuously differentiable function, there exists a strictly increasing function $\alpha(\cdot) \in C(\mathbb{R}^+;\mathbb{R})$ with $\alpha(0) = 0$ and a positive number σ such that:

$$V(x) \geq \alpha(x) \quad \text{for all} \quad x \in \bar{\mathbb{B}}_\sigma.$$

Without loss of generality, let $0 < \epsilon < \sigma$ and put $c = \alpha(\epsilon)$. According to the positive definiteness of V, there exists a constant $\eta > 0$ such that $\mathbb{B}_\eta \subset \Omega^0_c = \{x \in \mathbb{R}^n : V(x) < c\}$. Now we let $\sigma = \min\{\epsilon, \eta\}$. Take $\xi \in \mathbb{B}_\delta$ and $x(t;\xi)$ is a solution of (2.1) satisfying the initial condition $x(0) = \xi$. Suppose that there exists $t_1 \geq 0$ such that $\|x(t_1;\xi)\| \geq \epsilon$. Since $x(\cdot;\xi)$ is continuous, we may find some t^*satisfying: $\|x(t^*;\xi)\| = \epsilon$. Then, $V(x(t^*;\xi)) \geq \alpha(\|x(t^*;\xi)\|) = \alpha(\epsilon)$. On the other hand, V is decreasing along the trajectory on the time interval $[0,t^*]$ due to Remark 2.1 and the fact that $V^*(t,x) \geq 0$ for all $x \in \bar{\mathcal{B}}_\sigma$. Hence, we have $V(x(t^*;\xi)) \leq V(\xi) < c = \alpha(\epsilon)$. Our proof is finished by the contradiction. □

Theorem 2.2. *Let (A1) and (A2) hold. If there exists a Lyapunov function V for problem* (2.1) *such that $\dot{V}^*_\Phi(t,x) \leq -\lambda V(x)$ for all $x \in \bar{\mathbb{B}}_\sigma$ and for some $\lambda > 0$, for a.e. $t \geq 0$. Then the trivial solution is asymptotic stable.*

Proof. By Theorem 2.1, the trivial solution is stable. Hence, we can choose the number $\delta > 0$ such that for all $\xi \in \mathbb{R}^n$ and $\|\xi\| < \delta$, we have $x(t;\xi) \in \bar{\mathbb{B}}_\sigma$ for every $t \geq 0$. On the other hand, we have $\frac{d}{dt}V(x(t)) \in \dot{V}_\Phi(t, x(t))$ a.e. $t \geq 0$ and $\dot{V}^*_\Phi(t, x) \leq V(x)$ for all $x \in \bar{\mathbb{B}}_\sigma$. Then, we have:

$$\frac{d}{dt}V(x(t)) \leq -\lambda V(x(t)), \text{ a.e. } t \geq 0.$$

Taking the integration of both sides of this inequality, we obtain:

$$V(x(t)) \leq V(\xi)e^{-\lambda t}, \quad t \geq 0.$$

Therefore:

$$0 \leq \alpha(\|x(t)\|) \leq V(\xi)e^{-\lambda t}, \quad t \geq 0.$$

It follows from the fact that $\alpha(\cdot)$ is strictly increasing, we obtain:

$$\lim \sup_{t\to+\infty} \|x(t)\| = 0.$$

Then,

$$\lim_{t\to+\infty} \|x(t)\| = 0,$$

which leads to the result of the theorem. □

2.2. The invariance principle

We will generalize LaSalle's invariance principle to prove the asymptotic stability of the trivial solution. First, we recall some definitions and properties. Let $\xi \in \mathbb{R}^n$ and $x(t;\xi)$ be a solution of (2.1), denote the orbit of x by:

$$\gamma(x) = \{x(t;\xi) : t \geq 0\} \subset \mathbb{R}^n,$$

and the limit set of x by:

$$\Lambda(x) = \{p \in \mathbb{R}^n : \exists\{t_i\}, t_i \to +\infty \text{ as } i \to +\infty \text{ and } x(t_i;\xi) \to p\}.$$

Definition 2.3. *A set $S \subset \mathbb{R}^n$ is said weakly invariant if and only if for $\xi \in S$, there exists a solution of* (2.1) *starting at ξ contained in S. It is said to be invariant if and only if for $\xi \in S$, all solutions of* (2.1) *starting at ξ are contained in S.*

Assume that $\Phi(\cdot,\cdot)$ is *weakly asymptotically autonomous*, i.e. there exists $\Phi^* : x \to \Phi^*(x) \subset \mathbb{R}^n$ such that Φ^* is u.s.c and takes non-empty convex compact values such that for any compact $C \subset \mathbb{R}^n$, and any $\epsilon > 0$, there exists a $T \geq 0$ satisfying

$$\operatorname{ess\,sup}_{t\geq T} dist_H(\Phi(t, y), \Phi^*(y)) < \epsilon, \forall y \in C, \tag{2.2}$$

where $dist_H$ denotes the Hausdorff semidistance between two subsets. Then, the weakly-limiting DI is

$$x'(t) \in \Phi^*(x(t)). \tag{2.3}$$

We define a multivalued map $\Phi(\cdot,\cdot)$ is *strongly asymptotically autonomous*, if there exists $\hat{\mathbf{\Phi}}^*$ such that $\hat{\mathbf{\Phi}}^*$ is u.s.c and takes non-empty convex compact values such that for any compact $C \subset \mathbb{R}^n$, there exists $T \geq 0$ satisfying

$$\Phi(t,x) \subset \hat{\mathbf{\Phi}}^*(x), \forall t \geq T. \tag{2.4}$$

Then, the strongly-limiting DI is

$$x'(t) \in \hat{\mathbf{\Phi}}^*(x(t)). \tag{2.5}$$

We recall that

$$\dot{V}_{\Phi^*}(y) := \{\langle V'(y), \omega\rangle : \omega \in \Phi^*(y)\},\ \dot{V}^*_{\Phi^*}(y) = \sup_{\omega\in\Phi^*(y)} \langle V'(y), \omega\rangle.$$

In the case Φ is a singleton, H. Logemann and E. P. Ryan [14] proved the weak invariance with respect to the associated autonomous inclusions (2.3) of the $w-$limit set $\Lambda(x)$ whenever $x(\cdot)$ is a solution of converting differential inclusion (2.1). Here, we have such a situation in the multivalued case.

Lemma 2.1. *If $x(\cdot)$ is a bounded solution of* (2.1)*, then the limit set $\Lambda(x)$ of $x(\cdot)$ is non-empty, compact and connected, is approached by x and is weakly invariant with respect to* (2.3)*.*

Proof. The argument to prove this lemma based on [14, Proposition 4.1], where we replace the limiting mapping f_n by

$$f_n(t) := \mathrm{dist}_H(\Phi(t+t_n, x_n(t)), \Phi^*(x_n(t))) + \frac{1}{n},\ \forall t \geq 0.$$

□

Theorem 2.3. *(Invariance theorem) Suppose that there exist a function $V \in C^1(\mathbb{R}^n, \mathbb{R})$ such that $\dot{V}^*_{\Phi}(t,y) \leq 0$ for a.e. $t \geq 0$ and $y \in \mathbb{R}^n$. Let Ω be a compact invariant subset with respect to* (2.1) *of $\mathbb{R}^n, \xi \in \Omega$ and $x(\cdot;\xi)$ is a solution of* (2.1)*. Let $\mathcal{Z} = \{y \in \mathbb{R}^n : 0 \in \dot{V}_{\Phi^*}(y)\}$ and $\mathcal{M}$ be the largest weakly invariant subset with respect to* (2.3) *in the closure of $\mathcal{Z}$ then:*

$$\lim_{t\to+\infty} \mathit{dist}(x(t;\xi), \mathcal{M}) = 0.$$

Proof. Because $\xi \in \Omega$ and Ω is invariant, we have $\gamma(x) \in \Omega$. Therefore, $\gamma(x)$ is bounded and by Lemma 2.1, we obtain:

$$\lim_{t\to+\infty} \text{dist}(x(t;\xi), \Lambda(x)) = 0.$$

Thus, it is enough to prove that $\Lambda(x) \subset \bar{\mathcal{Z}}$ due to the weak invariance with respect to (2.3) of $\Lambda(x)$. Note that the function $V(\cdot)$ is of C^1 class, it is bounded on the compact set Ω. We imply that $V(x(\cdot))$ is decreasing on $\mathbb{R}^+$ since $\dot{V}^*_\Phi(t, y) \leq 0$ for a.e. $t \geq 0$ and $y \in \mathbb{R}^n$. Therefore, there exists a real number k such that $\lim_{t\to+\infty} V(x(t;\xi)) = k$. For each $p \in \Lambda(x)$, there exist $\{t_i\}$, $t_i \to +\infty$ as $i \to +\infty$ and $x(t_i;\xi) \to p$. Then, $V(p) = k$ due to the continuity of $V(\cdot)$. Hence, $V(p) = k$ for all $p \in \Lambda(x)$. Let $z \in \Lambda(x)$. Since $\Lambda(x)$ is weakly invariant with respect to (2.3), there exists a solution $\phi(t;z)$ of (2.3) lying in $\Lambda(x)$. Therefore: $V(\phi(t;z)) = k$, for all $t \geq 0$ which implies:

$$0 = \frac{d}{dt}V(\phi(t;z)) \in \dot{V}_{\Phi^*}(\phi(t,z)),$$

for almost all $t \geq 0$. Hence, we have

$$\phi(t;z) \in \mathcal{Z}$$

for almost all $t \geq 0$. Since $\phi(\cdot;z)$ is continuous, we obtain:

$$z = \phi(0;z) \in \bar{\mathcal{Z}},$$

and the result follows. □

3. Application in differential variational inequalities

In (1.1) - (1.3), we suppose that A, F, G satisfy the assumptions as follows.

(A) A is k-Lipschitz (not neccessary be a linear operator).

$$\|B(u)v\| \leq \eta_B(\|u\| + \|v\|), \text{ for some } \eta_B > 0, \forall u \in \mathbb{R}^n, \forall v \in K.$$

(F) $F : \mathbb{R}^n \to \mathbb{R}^m$ is continuous and $\|F(u)\| \leq \eta_F, \forall u \in \mathbb{R}^n$, $\eta_F > 0$.

(G) $G : K \to \mathbb{R}^m$ is a continuous function such that

(1) G is monotone on K, i.e.

$$\langle u - v, G(u) - G(v)\rangle \geq 0, \forall u, v \in K;$$

(2) G is coercive on K, i.e. there exists $v_0 \in K$ such that

$$\lim_{v\in K, \|v\|\to\infty} \frac{\langle v - v_0, G(v)\rangle}{\|v\|} = +\infty.$$

We have the following lemma.

Lemma 3.1. *Suppose that the assumptions* $(\mathbf{F})$ *and* $(\mathbf{G})$ *are satisfied. Then for each* $z \in \mathbb{R}^m$*, the solution set* $SOL(K, z + G(\cdot))$ *is nonempty, convex and closed. Moreover, there exists* $\rho > 0$ *such that*

$$\|SOL(K, z + G(\cdot))\| := \sup\{\|u\| : u \in SOL(K, z + G(\cdot))\} \leq \rho(1 + \|z\|). \tag{3.1}$$

We consider the nonautonomous DVI given by (1.1) - (1.3) with following assumptions:

(H1) A, F, G satisfy **(A)**, **(F)**, **(G)**.

(H2) $B : \mathbb{R}^+ \times \mathbb{R}^n \times K \to \mathcal{K}_V(\mathbb{R}^n)$ is *strongly asymptotically autonomous*, i.e. there exists $B^* : (y, u) \to B^*(y, u) \subset \mathbb{R}^n$ is u.s.c and takes non-empty convex compact values such that for all compact $C \subset \mathbb{R}^n$, $D \subset \mathbb{R}^m$, there exists $T \geq 0$ satisfying

$$B(t, y, u) \subset B^*(y, u), \forall t \geq T,$$

where we denote the collection of nonempty, compact, convex, subsets of $\mathbb{R}^n$ by $\mathcal{K}_V(\mathbb{R}^n)$.

Then the converting DI is

$$x'(t) \in Ax(t) + B(t, x(t), SOL(K, F(x(t)) + G(\cdot))) := \Phi(t, x(t)), \tag{3.2}$$
$$x(0) = \xi, \tag{3.3}$$

and the strongly limiting DI is

$$x'(t) \in Ax(t) + B^*(x(t), SOL(K, F(x(t)) + G(\cdot))) := \Phi^*(x(t)), \tag{3.4}$$
$$x(0) = \xi. \tag{3.5}$$

Theorem 3.1. *Suppose that* **(H1)** *and* **(H2)** *hold. Then for each* $\xi \in \mathbb{R}^n$*, the problem* (1.1) - (1.3) *has an absolutely continuous solution.*

Corollary 3.1. *If* $B(t, 0, 0) = 0, F(0) = 0$ *and* $\dot{V}^*_{\Phi^*}(x) < 0 \;\forall x \in \mathbb{B}_\delta \setminus \{0\}$*, for some* $\delta > 0$*,* $V_{\Phi^*}(0) = 0$*. Then the trivial solution is asymptotically stable.*

- Example 1

Let $K = \mathcal{C}$ be a cone in $\mathbb{R}^2$. Consider the two-dimensional nonautonomous DVI

$$\dot{x}(t) = [A(t) + B(t)]x(t) + u(t), \tag{3.6}$$
$$\mathcal{C} \ni u(t) \perp C(x(t)) + D(u(t)) \in \mathcal{C}^*, \tag{3.7}$$

where $A(t) = \begin{bmatrix} f_1(t) & f_3(t) \\ -f_3(t) & f_2(t) \end{bmatrix}$, f_i, B, h are of class L^1_{loc}; C, D are linear operators. In what follows, we assume

(i) f_1, f_2, f_3 approach compact intervals I_1, I_2, I_3 with $I_1, I_2 \subset (-\infty, -\delta)$ for some $\delta > 0$, by the mean $f_i(t) \in I_i, \forall t \geq T$, where $T > 0$ large enough.

(ii) $\|B(\cdot)\| \in L^1$ with $\|B(t)\| \to 0$ as $t \to +\infty$,

(iii) $u^T D u \geq \eta_D \|u\|^2$ for some $\eta_D > 0$.

We have the following result.

Theorem 3.2. *For each initial value $x(0) = \xi$, there exists a unique solution of* (3.6)-(3.7). *If $\eta_D \delta > \|C\|$, then the trivial solution is globally asymptotically stable.*

Proof. By D is linear and satisfies the coercive property (iii), we have $SOL(\mathcal{C}; z + D(\cdot))$ is a singleton and Lipschitz with the Lipchizian-constant $\frac{1}{\eta_D}$. Thus, the existence and uniqueness of a solution of (3.6) - (3.7) follows.

We can denote

$$\text{SOL}(\mathcal{C}, z + D(\cdot)) = (\psi_1(z), \psi_2(z)) \in \mathbb{R}^2.$$

Then we have $SOL(\mathcal{C}; C, D)(x) := SOL(\mathcal{C}, C(x) + D(\cdot)), \forall x \in \mathbb{R}^n$. The limiting differential inclusion in this case is $x'(t) \in \Phi^*(x(t))$, where

$$\Phi^*(x) = \Phi^*(x_1, x_2) = \{(\alpha_1 x_1 + \alpha_3 x_2 + \psi_1(Cx), -\alpha_3 x_1 + \alpha_2 x_2 + \psi_2(Cx)) : \alpha_i \in I_i\}.$$

Choose $V(x) = \frac{1}{2}\|x\|^2$, we have to prove that $\dot{V}^*(x) < 0, V^*(x) = 0$ for all $x \in \mathbb{R}^2$. In fact, we have

$$\begin{aligned}
\langle V'(x), \Phi^*(x)\rangle &= \langle (x_1, x_2), (\alpha_1 x_1 + \alpha_3 x_2 + \psi_1(Cx), -\alpha_3 x_1 + \alpha_2 x_2 + \psi_2(Cx))\rangle \\
&= \alpha_1 x_1^2 + \alpha_2 x_2^2 + x_1 \psi_1(Cx) + x_2 \psi_2(Cx) \\
&\leq -\delta(x_1^2 + x_2^2) + \|x\|\|\psi(Cx)\| \\
&\leq -\delta\|x\|^2(1 - \frac{\|C\|}{\eta_D \delta}) < 0, \forall x \neq 0.
\end{aligned}$$

for all $x \in \mathbb{R}^2$. By the Corollary 3.1, the proof is complete. □

- Example 2

We consider a differential variational inequality of the inclusion form as follows

$$x'(t) = Ax(t) + \xi u(t), t > 0 \quad (3.8)$$

$$\xi \in [f_1(t), f_2(t)], \quad (3.9)$$

$$\mathcal{C} \ni u(t) \perp C(x(t)) + D(u(t)) \in \mathcal{C}^*, \quad (3.10)$$

$$x(0) = x_0, \tag{3.11}$$

where $\mathcal{C}$ is a cone in $\mathbb{R}^m$, then we have the converting differential inclusion

$$x'(t) \in Ax(t) + [f_1(t), f_2(t)]SOL(\mathcal{C}; C, D)(x(t)), t > 0.$$

Suppose that

(iv) A is a linear operator.

(v) $f_1(\cdot), f_2(\cdot) : \mathbb{R}^+ \to \mathbb{R}$ are continuous and satisfying the following properties

$$f_1(t) \downarrow a; f_2(t) \uparrow b \text{ as } t \to +\infty,$$

for some $a < b$.

(vi) C, D are assumed as Example 2.

We obtain that the strongly limiting differential inclusion is:

$$x'(t) \in Ax(t) + [a, b]\Psi(Cx(t)), t > 0,$$

We conclude with the statement on the stability of the solution $x = 0$.

Theorem 3.3. *For each initial value $x(0) = \xi$, there exists a solution of* (3.8)-(3.11). *If $\eta > \dfrac{a\|C\|}{\eta_D}$, then the trivial solution is globally asymptotic stable.*

REFERENCES

[1] Pang JS & Stewart DE, (2018). Differential variational inequalities. *Mathematical Programming*, 113, 345-424.

[2] Avgerinos EP & Papageorgiou NS, (1997). Differential variational inequalities in $\mathbf{R}^{\mathbb{N}}$. *Indian Journal of Pure and Applied Mathematics*, 28(9), 1267-1287.

[3] Camlibel MK, Pang JS & Stone RE, (2006). Lyapunov stability of complementarity and extended systems. *SIAM Journal on Optimization*, 17(4), 1056-1101.

[4] Chen XJ & Wang ZY, (2014). Differential variational inequality approach to dynamic games with shared constraints. *Mathematical Programming*, 146, 379–408.

[5] Raghunathan AU, Pérez-Correa JR, Agosin E & Biegler LT, (2006). Parameter estimation in m etabolic flux balance models for batchfermentation-formulation and solution using differential variational inequalities. *Annals of Operations Research*, 148, 251–270.

[6] Nguyen TVA & Tran DK, (2015). Asymptotic behavior of solutions to a class of differential variational inequalities. *Annales Polonici Mathematici*, 147-164.

[7] Nguyen TVA, (2022). On periodic solutions to a class of delay differential variational inequalities. *Evolution Equations and Control Theory*, 11(4), 1309–1329.

[8] Liu Z, Nguyen VL & Valeri O, (2013). Existence and global bifurcation of periodic solutions to a class of differential variational inequalities. *International Journal of Bifurcation and Chaos*, 23(7), 1350125, 1-10.

[9] Nguyen VL, (2015). On two-parameter global bifurcation of periodic solutions to a class of differential variational inequalities. *Nonlinear Analysis*, 122 (2015), 83–99.

[10] LaSalle JP, (1976). *Stability theory and invariance principles, Dynamical Systems*. Academic Press, New York, I, 211-222.

[11] Adly S & Le BK, (2014). Stability and invariance results for a class of non-monotone set-valued Lur'e dynamical systems. *Applicable Analysis*, 5(278) 1087–1105.

[12] Finogenko IA, (2014). Limit differential inclusions and the invariance principle for nonautonomous systems. *Siberian Mathematical Journal*, 55:2, 372–386.

[13] Finogenko IA, (2016). The invariance principle for nonautonomous differential equations with discontinuous right-hand side. *Sibirskii Matematicheskii Zhurnal*, 57:4, 913–927.

[14] Logemann H & Ryan EP, (2003). Non-autonomous systems: asymptotic behaviour and weak invariance principles. *Journal of Differential Equations*, 189(2) 440–460.

[15] Filippov AF, (1988). Differential equations with discontinuous right-hand side. *Dordrecht: Kluwer Academic*.