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ROOTS OF COMPLEX HARMONIC POLYNOMIALS IN ONE VARIABLE

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Abstract. In this paper, we establish some bounds on the moduli of the zeros of complex harmonic polynomials in one variable. We also present explicit examples that illustrate these results. Our findings contribute to a deeper understanding of zeros of complex harmonic polynomials in one variable.
Keywords: harmonic functions, harmonic polynomials, complex polynomials.

1. Introduction

Complex polynomials in one variable play a central role in mathematics, serving as fundamental tools in various areas such as complex analysis, algebraic geometry, or dynamical systems due to their rich structures and wide-ranging applications. For example, the fundamental theorem of algebra tells us that a complex polynomial of degree n has exactly n roots, counting with multiplicities. In general, it is quite difficult to compute exact values of polynomial roots. Therefore, rather than seeking exact values, it is natural to focus on estimating bounds for the size of the roots. A significant result in this direction was established by Cauchy (see Theorem 2.1 in the next section). For more refined estimates of the number of roots of complex polynomials, the reader is referred to Chapter 1 in [1].

The primary focus of this note is regarding harmonic polynomials, which are complex-valued functions of the form

$$P(z) = h(z) + \overline{g(z)},$$

with h and g are analytic. This type of polynomials arises naturally in various problems in complex analysis and mathematical physics. Unlike analytic polynomials, harmonic polynomials may exhibit more intricate zero structures, including a higher number of isolated zeros and more complex geometric arrangements. For more information on harmonic functions, the reader may consult [2].

A fundamental question concerns the number and location of zeros of such functions. In particular, using the argument principle for harmonic functions in [3], it was shown in the same paper that if $\deg(h) = n > m = \deg(g)$, then F have at most n^2 isolated zeros in the complex plane. This bound is known to be sharp, but only for certain specific configurations.

Despite the simple appearance of harmonic polynomials, determining their zero sets remains a challenging and largely open problem. Many researchers have focused on understanding how the algebraic form of h and g—especially the degrees and coefficients—influences the number and distribution of zeros.

A particularly tractable and intriguing class of harmonic polynomials is formed by those where g is a monomial, yielding expressions like

$$p(z) = z^n + c\overline{z}^k - 1,$$

with $n > k \geq 0$, and $c \in \mathbb{C} \setminus \{0\}$. These polynomials serve as natural test cases to explore extremal behavior, providing insight into broader conjectures and bounds on the number of zeros. Some partial results on bounding the number of roots of such polynomials can be found in [4] and [5].

This paper concerns with the geometric constraints of such polynomials, specifically on locating annular regions that contain all their zeros. We aim to provide explicit bounds in terms of the coefficients and degrees, thereby contributing to a deeper understanding of their complex structure.

2. Main results

To motivate our research, we will recall here the classical Cauchy bound of roots of complex polynomials. A proof of this fact can be found, for ecample, in Theorem 1.1.3 in [1].

Theorem 2.1. *Let P be a complex polynomial given by*

$$P(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0,$$

with $a_n \neq 0$. Then every root $z \in \mathbb{C}$ of the equation $P(z) = 0$ satisfies

$$|z| \leq R := 1 + \max_{0 \leq k \leq n-1} \left| \frac{a_k}{a_n} \right|.$$

Now we come to the first main result of this note, which generalizes the Cauchy bound to harmonic polynomials.

Theorem 2.2. *Let*

$$P(z) = \sum_{k=0}^{n} a_k z^k + \sum_{k=0}^{m} b_k \overline{z}^k,$$

where $a_k, b_k \in \mathbb{C}$*, with* $a_n \neq 0$ *and* $n \geq m$*. Then all zeros of* P *lie in the closed disk centered at the origin of radius*

$$R := \max(1, r), \tag{2.1}$$

where $r \neq 1$ *is the positive real root of the equation*

$$x^{n+1} - (1 + M)x^n + M = 0, \tag{2.2}$$

with

$$M := \max_{0 \leq k \leq n-1} \frac{|a_k| + |b_k|}{|a_n|}. \tag{2.3}$$

Proof. Let $z \in \mathbb{C}$ be a point such that $|z| > R$, where R will be determined to ensure that $P(z) \neq 0$. We aim to show that $P(z) \neq 0$ for all such z, so that all zeros lie within the closed disk of radius R.

We rewrite $P(z)$ as

$$P(z) = a_n z^n + \sum_{k=0}^{n-1} a_k z^k + \sum_{k=0}^{m} b_k \overline{z}^k.$$

Take modulus on both sides we get

$$\begin{aligned} |P(z)| &\geq |a_n z^n| - \left| \sum_{k=0}^{n-1} a_k z^k \right| - \left| \sum_{k=0}^{m} b_k \overline{z}^k \right| \\ &\geq |a_n||z|^n - \sum_{k=0}^{n-1} |a_k||z|^k - \sum_{k=0}^{m} |b_k||z|^k. \end{aligned}$$

Combining the two sums over $|z|^k$, we get

$$|P(z)| \geq |a_n||z|^n - \sum_{k=0}^{n-1} (|a_k| + |b_k|)|z|^k.$$

Define the function

$$Q(r) := r^n - \sum_{k=0}^{n-1} \frac{|a_k| + |b_k|}{|a_n|} r^k, \quad \text{for } r = |z| > 0. \tag{2.4}$$

Let

$$M := \max_{0 \leq k \leq n-1} \frac{|a_k| + |b_k|}{|a_n|}.$$

Then it is clear that

$$Q(r) \geq r^n - M \sum_{k=0}^{n-1} r^k = r^n - M \cdot \frac{r^n - 1}{r - 1}, \quad \text{for } r > 1.$$

To simplify the analysis, consider the polynomial

$$f(x) := x^{n+1} - (1+M)x^n + M. \tag{2.5}$$

Observe that

$$f(x) = (x-1)\left(x^n - M \cdot \frac{x^n - 1}{x-1}\right) = (x-1)Q(x), \quad \text{for } x > 1.$$

Since $f(x) \to +\infty$ as $x \to \infty$ and $f(1) = -M < 0$, the Intermediate Value Theorem guarantees a unique real root $r > 1$ such that $f(r) = 0$, equivalently $Q(r) = 0$.

Hence, for all $|z| > r$, we have $Q(|z|) > 0 \Rightarrow |P(z)| > 0$, and thus z cannot be a root.

Therefore, all roots of $P(z)$ lie in the closed disk centered at the origin of radius

$$R := \max(1, r),$$

which completes the proof. □

Corollary 2.1. *Let*

$$p(z) = z^n + c\,\overline{z}^k - 1,$$

where $n > k \geq 0$*,* $c \in \mathbb{C} \setminus \{0\}$*. Then all the roots of* p *lie in the closed disk centered at the origin of radius*

$$R = 2 + |c|. \tag{2.6}$$

Proof. We apply Theorem 2.2 to the harmonic polynomial

$$p(z) = z^n - 1 + c\,\overline{z}^k = H(z) + G(z),$$

where $H(z) = z^n - 1$ and $G(z) = c\,\overline{z}^k$. Clearly, $\deg(H) = n$, and $a_n = 1$, $a_0 = -1$, $b_k = c$, all other coefficients zero.

Then for $0 \leq j \leq n-1$, we have

$$M := \max_{0 \leq j \leq n-1} \frac{|a_j| + |b_j|}{|a_n|} = 1 + |c|.$$

Theorem 2.2 tells us that all roots lie within the closed disk of radius equal to the largest positive real root of

$$x^{n+1} - (1+M)x^n + M = 0.$$

It can be verified that $r = 1 + M$ is an upper bound on that root. Indeed, by direct computations, we get

$$x^{n+1} - (1+M)x^n + M = 0 \Rightarrow x^{n+1} = (1+M)x^n - M \Rightarrow x^{n+1} \leq (1+M)x^n.$$

Since $x > 0$ we can infer that

$$x \leq r := 1 + M.$$

Therefore, every root of p must lie in the closed disk centered at the origin with radius $R := \max(1, r) = 1 + M = 2 + |c|$. □

We move to another result on the location of roots for certain harmonic polynomials with conjugate terms.

Theorem 2.3. *Let*

$$p(z) = z^n + c\bar{z}^k - 1,$$

where $n \geq 3$, $1 \leq k \leq n-1$, $\gcd(n,k) = 1$, *and* $c \in \mathbb{C} \setminus \{0\}$. *Then, all the roots of* p *lie in an annular region centered at the origin, depending on the modulus and sign of* c, *as follows:*

(a) If $0 < c < 1$, *then every root* z *satisfies*

$$(1-c)^{1/(n-k)} < |z| < (1+c)^{1/(n-k)}. \tag{2.7}$$

(b) If $c > 1$ *and* $|z| \geq 1$, *then*

$$(c-1)^{1/(n-k)} \leq |z| \leq (1+c)^{1/(n-k)}. \tag{2.8}$$

(c) For general $c \in \mathbb{C}$, *if* $|z| \geq 1$, *then*

$$(|c|-1)^{1/(n-k)} \leq |z| \leq (1+|c|)^{1/(n-k)}. \tag{2.9}$$

Note that the assumption on $\gcd(n,k) = 1$ is imposed, but on the other hand, more refined information on the location of roots is obtained.

Proof. We treat each case separately.
Case (a): $0 < c < 1$

Let $z \in \mathbb{C}$ be a root of $p(z) = z^n + c\bar{z}^k - 1$. Taking moduli on both sides

$$|z^n + c\bar{z}^k| = 1.$$

Using the triangle inequality:

$$|z|^n - c|z|^k \leq 1 \leq |z|^n + c|z|^k.$$

Letting $r = |z|$, we get

$$\begin{aligned} r^n - cr^k < 1 &\Rightarrow r^{n-k} - c < r^{-k} \Rightarrow r^{n-k} < c + r^{-k}, \\ r^n + cr^k > 1 &\Rightarrow r^{n-k} + c > r^{-k} \Rightarrow r^{n-k} > -c + r^{-k}. \end{aligned}$$

By choosing r so that $r^{n-k} = 1 + c$ and $r^{n-k} = 1 - c$, we define the bounds

$$(1-c)^{1/(n-k)} < |z| < (1+c)^{1/(n-k)}.$$

This shows that the roots lie in a symmetric annulus.

To ensure all roots are captured and the region is not vacuous, note that since $\gcd(n, k) = 1$, the equation defines a harmonic function with isolated and finite number of roots. This follows from the argument principle for harmonic functions (in [3]), showing that exactly n roots exist when $0 < c < 1$.

Case (b): $c > 1$ and $|z| \geq 1$

As before, we start with

$$|z^n + c\bar{z}^k| = 1.$$

By applying the reverse triangle inequality, we get

$$|z|^n - c|z|^k \leq 1 \leq |z|^n + c|z|^k.$$

Let $r = |z| \geq 1$, we proceed as in the analytic case, then

$$r^{n-k} \leq 1 + c \Rightarrow r \leq (1 + c)^{1/(n-k)}.$$

Also,

$$|z^n + c\bar{z}^k| \geq ||z|^n - c|z|^k| \geq 1 \Rightarrow r^{n-k} \geq c - 1 \Rightarrow r \geq (c - 1)^{1/(n-k)}.$$

Therefore, any root with $|z| \geq 1$ must lie in the annulus

$$(c - 1)^{1/(n-k)} \leq |z| \leq (1 + c)^{1/(n-k)}.$$

Case (c): General complex $c \in \mathbb{C}$, with $|z| \geq 1$

We generalize part (b) by replacing c with $|c|$. Since

$$|z^n + c\bar{z}^k| \leq |z|^n + |c||z|^k = |z|^k(|z|^{n-k} + |c|),$$

we derive

$$|z|^{n-k} \leq 1 + |c| \Rightarrow |z| \leq (1 + |c|)^{1/(n-k)}.$$

Similarly, by reverse inequality

$$|z^n + c\bar{z}^k| \geq \big||z|^n - |c||z|^k\big| \Rightarrow |z|^{n-k} \geq |c| - 1 \Rightarrow |z| \geq (|c| - 1)^{1/(n-k)}.$$

This confirms that any root z with $|z| \geq 1$ must satisfy

$$(|c| - 1)^{1/(n-k)} \leq |z| \leq (1 + |c|)^{1/(n-k)}.$$

By summing up all these arguments, we complete the proof of the theorem. □

We apply Theorem 2.3 to the harmonic polynomial

$$p(z) = z^8 + \overline{z}^3 - 1.$$

Then, we have

$$n = 8, \quad k = 3, \quad c = 1,$$

so that $n - k = 5$ and $c = 1$. This case corresponds to the boundary between Theorem 3.3(a) and (b).

Thus, for any root z of p with $|z| \geq 1$, we obtain the upper bound

$$|z| \leq (1 + c)^{1/(n-k)} = (2)^{1/5} \approx 1.1487.$$

On the other hand, we let $r_0 \approx 0.872$ be the positive root of the equation

$$x^8 + x^3 = 1.$$

By the triangle inequality, we see that any root z of p must satisfy $|z| > r_0$. Thus the annular $\{0.872 < |z| < 1.1487\}$ contains all roots of $p(z) = 0$.

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