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ULTIMATE BOUNDS OF RAPIDLY TIME-VARYING SYSTEMS: AN AVERAGING APPROACH

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Abstract. This paper addresses the problem of state bounding for rapidly time-varying linear systems with bounded disturbances. Based on an averaging method and Lyapunov scheme, we derive an explicit bound on the time-scale parameter for which an explicit form of asymptotic bound of solutions is obtained. An application in fast switching systems is presented to illustrate the effectiveness of the analysis results.
Keywords: ultimate bounds, rapidly time-varying systems, averaging method.

1. Introduction

Exogenous disturbances encountered in data processing, approximations or measurement errors are unavoidable in systems and control engineering [1]. In the effect of disturbances, the system behavior is typically unpredictable or even unstable. Thus, it is important to determine the set of reachable states starting from the origin [1], [2] or, more general, is to determine the absorbing set of states under bounded disturbances [3]. For linear time-invariant systems, variant schemes of the Lyapunov function method are employed to derive testable conditions in the form of linear matrix inequalities (LMIs) to ensure the existence of such an absorbing set (see, [3], [4] and the references therein). Unfortunately, this method cannot be extended to nonautonomous systems [5].

On the other hand, dynamical systems with almost periodic parameters are essential to physics and engineering according to numerous applications in vibrational control or power systems [6]. Such systems often involve components subject to multiple time-scales, which raises requirements in the analysis of systems with rapidly time-varying coefficients. The averaging method is one of the most important perturbation-based techniques for the study of stability of systems with oscillatory inputs [7]. In this paper, we develop the averaging method for the problem of state bounding of rapidly time-varying linear systems in the presence of bounded disturbances. It will be shown that asymptotic stability of the averaged system guarantees the existence of an absorbing set of the original rapidly-varying system for small enough values of the time-scale parameter. More precisely, based on an averaging assumption and Lyapunov scheme, we derive an explicit bound on the time-scale parameter for which an explicit form of the asymptotic bound of solutions is obtained. We illustrate the effectiveness of the analysis results via a numerical application in fast switching systems.

2. Preliminaries

We begin with the following scalar differential equation

$$\begin{aligned} x'(t) &= ax(t) + w(t),\ t \geq 0, \\ x(0) &= x_0, \end{aligned} \tag{2.1}$$

where a is a real number and $w \in L_\infty[0, \infty)$ with

$$\|w\|_{L_\infty} := \sup_{t \geq 0} |w(t)| < \infty. \tag{2.2}$$

Typically, w represents a type of exogenous bounded disturbances to equation (2.1). The solution $x(t) = x(t, x_0, w)$ of (2.1) can be represented as

$$x(t) = e^{at}\left(x_0 + \int_0^t e^{-as} w(s) ds\right),\ t \geq 0. \tag{2.3}$$

It can be verified from (2.3) that the solution $x(t)$ is ultimately bounded, that is, $\limsup_{t\to\infty} |x(t)| < \infty$, for any exogenous w satisfying (2.2) if and only if $a < 0$. Moreover, in such a case, we get the following *uniform ultimate bound* (UUB) for all solutions of (2.1) subject to (2.2)

$$\limsup_{t\to\infty} |x(t)| \leq \frac{\|w\|_{L_\infty}}{-a}. \tag{2.4}$$

More generally, for the case of multi-dimensional systems, equation (2.1) becomes

$$\begin{aligned} x'(t) &= Ax(t) + w(t),\ t \geq 0, \\ x(0) &= x_0 \in \mathbb{R}^n, \end{aligned} \tag{2.5}$$

where $x(t) \in \mathbb{R}^n$ is the state vector, $w(t) \in \mathbb{R}^n$ is the external input, and $A \in \mathbb{R}^{n\times n}$ is a given real matrix.

System (2.5) is said to be *ultimately bounded* if there exists a constant (referred as a threshold factor of the system states) $\beta > 0$ such that any solution $x(t) = x(t, x_0, w)$ of (2.5) satisfies

$$\limsup_{t\to\infty} \|x(t, x_0, w)\| \leq \beta \|w\|_{L_\infty}, \tag{2.6}$$

where $\|w\|_{L_\infty} = \sup_{t\geq 0}\|w(t)\|$. For a given threshold $\overline{w} \geq 0$ of external disturbances, it can be seen that $\limsup_{t\to\infty}\|x(t,x_0,w)\| \leq \beta\overline{w}$ holds for all $x_0 \in \mathbb{R}^n$ and $w \in L_\infty(\mathbb{R}_+,\mathbb{R}^n)$ with $\|w\|_{L_\infty} \leq \overline{w}$. In this meaning, the constant $\beta\overline{w}$ determines a UUB of system (2.5). Thus, the estimate in (2.6) plays an important role in determining the effect of bounded exogenous disturbances on the system states.

Assume that the matrix A is Hurwitz (i.e. all eigenvalues of A are located on the left of the complex plane). Then, there exists a symmetric positive definite matrix, written as $P \in \mathbb{S}_n^+$, such that $A^\top P + PA < 0$. Moreover, there exists a scalar $\alpha > 0$ such that

$$A^\top P + PA + 2\alpha P < 0. \tag{2.7}$$

Based on (2.7), we have the following result.

Proposition 2.1. *An UUB of system* (2.5) *can be obtained as*

$$\limsup_{t\to\infty}\|x(t,x_0,w)\| \leq \frac{\sqrt{\nu(P)}}{\alpha}\|w\|_{L_\infty}, \tag{2.8}$$

whee $\nu(P) = \lambda_{\max}(P)/\lambda_{\min}(P)$ is the conditional number of the matrix P.

It is important to note that the proof of this result cannot be directly derived as (2.4). To make the paper self contain, we give a brief proof as follows. Let P and α be determined by (2.7) and consider the following Lyapunov function $V(t) = \|x(t)\|_P^2 = x^\top(t)Px(t)$. It is fact that for any vectors $u, v \in \mathbb{R}^n$ and positive scalar δ, the following Cauchy-type inequality holds

$$2u^\top v \leq \delta\|u\|^2 + \frac{1}{\delta}\|v\|^2.$$

By using this fact, the derivative of $V(t)$ along any state trajectory $x = x(t)$ of (2.5) is obtained as

$$\begin{aligned} V'(t) + \alpha V(t) &= x^\top(t)(A^\top P + PA + \alpha P)x(t) + 2x^\top(t)Pw(t) \\ &\leq -\alpha\|x(t)\|_P^2 + \delta\|Px(t)\|^2 + \frac{1}{\delta}\|w(t)\|^2 \\ &\leq -(\alpha - \delta\lambda_{\max}(P))\|x(t)\|_P^2 + \frac{1}{\delta}\|w(t)\|^2. \end{aligned} \tag{2.9}$$

By selecting $\delta = \alpha/\lambda_{\max}(P)$, it follows from (2.4) and (2.9) that

$$V(t) \leq V(0)e^{-\alpha t} + \frac{\lambda_{\max}(P)\|w\|_{L_\infty}^2}{\alpha^2}\left(1 - e^{-\alpha t}\right)$$

by which we readily obtain

$$\limsup_{t\to\infty} V(t) \leq \frac{\lambda_{\max}(P)\|w\|_{L_\infty}^2}{\alpha^2}.$$

This, in combination with the fact $V(t) \geq \lambda_{\min}(P)\|x(t)\|^2$, yields the desired estimation in (2.8).

Remark 2.1. *The aforementioned ultimate bounds fail to hold for time-varying systems. For a counterexample, we consider the following equation*

$$x'(t) = -\frac{1}{1+t}x(t) + w(t),\ t \geq 0. \tag{2.10}$$

It is clear that

$$a(t) = -\frac{1}{1+t} < 0$$

for all $t \geq 0$. *However, with constant exogenous* $w(t) = w \neq 0$, *we have*

$$x(t) = \frac{x_0}{1+t} + (-t + \ln(1+t))w.$$

Thus, $\limsup_{t\to\infty} |x(t)| = \infty$.

3. Rapidly time-varying systems

Consider the following fast-varying system

$$x'(t) = A(t/\epsilon)x(t) + w(t),\ t \geq 0, \tag{3.1}$$

where $x(t) \in \mathbb{R}^n$ is the state vector, $w(t) \in \mathbb{R}^n$ represents the external disturbance and $A : \mathbb{R}_+ \to \mathbb{R}^{n\times n}$ is a piecewise continuous, bounded matrix function. In system (3.1), the small parameter $0 < \epsilon \ll 1$ defines a fast time-scale.

Assumption (A1): There exist a $T > 0$ and a real matrix $\tilde{A} \in \mathbb{R}^{n\times n}$ such that

$$\frac{1}{T}\int_t^{t+T} A(s)ds = \tilde{A} + \Delta A(t), \tag{3.2}$$

where $\Delta A : \mathbb{R}_+ \to \mathbb{R}^{n\times n}$ is a matrix function satisfying

$$\delta_A = \sup_{t\geq 0} \|\Delta A(t)\| < \infty.$$

Our main aim in this paper is to derive a UUB for a rapidly time-varying system (3.1) subject to conditions that are imposed on $\tilde{A}$, δ_A, η_A, and T, ϵ, where $\eta_A = \sup_{t\geq 0} \|A(t)\|$.

3.1. Ultimate boundedness

Motivated by the transformation method proposed in [8], we define the matrix function

$$Z_\epsilon(t) = -\frac{1}{\epsilon T}\int_t^{t+\epsilon T} (t + \epsilon T - \theta)A(\theta/\epsilon)d\theta. \tag{3.3}$$

It is clear from (3.3) that the matrix function $Z_\epsilon(t)$ is piecewise continuous differentiable on $[0,\infty)$. Moreover, we have

$$\begin{aligned}\|Z_\epsilon(t)\| &\le \frac{1}{\epsilon T}\int_t^{t+\epsilon T}(t+\epsilon T-\theta)\|A(\theta/\epsilon)\|d\theta\\ &\le \frac{\eta_A}{\epsilon T}\int_t^{t+\epsilon T}(t+\epsilon T-\theta)d\theta = \frac{1}{2}\epsilon T\eta_A. \end{aligned}\tag{3.4}$$

Assumption (A2): The time-scale parameter ϵ satisfies

$$\epsilon\in(0,\epsilon_*),\ \text{where}\ \epsilon_* = \frac{2}{T\eta_A}.$$

According to (A2), we have

$$\|Z_\epsilon(t)\| \le \sigma \triangleq \frac{1}{2}\epsilon T\eta_A < 1.$$

Therefore, the time-dependent matrix $I - Z_\epsilon(t)$ is always invertible with

$$(I - Z_\epsilon(t))^{-1} = \sum_{k=0}^{\infty} Z_\epsilon^k(t).$$

Moreover, the following norm-estimate holds

$$\left\|(I - Z_\epsilon(t))^{-1}\right\| \le \frac{1}{1-\sigma},\ t\ge 0. \tag{3.5}$$

On the other hand, for a.e. $t>0$, it follows from (3.3) that

$$\begin{aligned}Z_\epsilon'(t) &= A(t/\epsilon) - \frac{1}{T}\int_{t/\epsilon}^{t/\epsilon+T} A(\theta)d\theta\\ &= A(t/\epsilon) - \left(\tilde{A} + \Delta A(t/\epsilon)\right).\end{aligned}$$

By substituting the last equality into (3.1), we get

$$x'(t) = \left(\tilde{A} + Z_\epsilon'(t) + \Delta A(t/\epsilon)\right)x(t) + w(t),\ t\ge 0. \tag{3.6}$$

In practice, it is hard to manipulate the time-dependent matrix term $Z_\epsilon'(t)$. To eliminate this term, we introduce the state transformation

$$z(t) = (I - Z_\epsilon(t))x(t),\ t\ge 0. \tag{3.7}$$

It follows from (3.7) that

$$x(t) = (I - Z_\epsilon(t))^{-1}z(t)$$

and, in combination with (3.6), we have

$$z'(t) = \left(\tilde{A} + G(t)\right) z(t) + H(t)w(t), \tag{3.8}$$

where

$$\begin{aligned} G(t) &= \left[\tilde{A}Z_\epsilon(t) - Z_\epsilon(t)A(t/\epsilon) + \Delta A(t/\epsilon)\right](I - Z_\epsilon(t))^{-1}, \\ H(t) &= I - Z_\epsilon(t). \end{aligned}$$

Note also from (3.8) that

$$\begin{aligned} \|G(t)\| &\le \frac{\sigma(\|\tilde{A}\| + \eta_A) + \delta_A}{1-\sigma} =: g_A, \\ \|H(t) &\le 1 + \sigma =: h_A. \end{aligned} \tag{3.9}$$

We are now in a position to present our main result as given in the following theorem.

Theorem 3.1. *Let Assumptions* (A1) *and* (A2) *hold. Assume that, for given matrix $\tilde{A}$ and constants ϵ, T, δ_A, η_A, there exist a symmetric positive definite matrix $P \in \mathbb{R}^{n\times n}$, positive scalars σ_1, σ_2, and σ_3 that satisfy the following linear matrix inequality*

$$\begin{pmatrix} \tilde{A}^\top P + P\tilde{A} + \sigma_1 P + \sigma_2 g_A^2 I & P & P \\ * & -\sigma_2 I & 0 \\ * & * & -\sigma_3 I \end{pmatrix} < 0. \tag{3.10}$$

Then, a UUB for the rapidly time-varying system (3.1) *can be obtained*

$$\limsup_{t\to\infty} \|x(t, x_0, w, \epsilon)\| \le \frac{2 + \epsilon T \eta_A}{2 - \epsilon T \eta_A} \sqrt{\frac{\sigma_3}{\sigma_1 \lambda_{\min}(P)}} \|w\|_{L_\infty}. \tag{3.11}$$

Proof. Consider the following Lyapunov function

$$V(t) = \|z(t)\|_P^2 = z^\top(t) P z(t).$$

Taking the derivative of $V(t)$ along state trajectory $z(t)$ of (3.8), we then obtain

$$\begin{aligned} V'(t) + \sigma_1 V(t) = z^\top(t)&\left(\tilde{A}^\top P + P\tilde{A} + \sigma_1 P\right) z(t) \\ &+ 2z^\top(t) P G(t) z(t) + 2z^\top(t) P H(t) w(t). \end{aligned} \tag{3.12}$$

By utilizing the Cauchy inequality to (3.12), we have

$$\begin{aligned} 2z^\top(t) P G(t) z(t) &\le \sigma_2^{-1} \|Pz(t)\|^2 + \sigma_2 \|G(t)z(t)\|^2 \\ &\le z^\top(t)\left(\sigma_2^{-1} P^2 + \sigma_2 g_A^2 I\right) z(t) \end{aligned}$$

and

$$\begin{aligned}2z^\top(t)PH(t)w(t) &\leq \sigma_3^{-1}\|Pz(t)\|^2 + \sigma_3\|H(t)w(t)\|^2\\ &\leq \sigma_3^{-1}z^\top(t)P^2z(t) + \sigma_3 h_A^2\|w(t)\|^2.\end{aligned}$$

Taking the above manipulations into account, from (3.12), we get

$$V'(t) + \sigma_1 V(t) \leq z^\top(t)\mathcal{M}z(t) + \sigma_3 h_A^2\|w(t)\|^2,\ t \geq 0, \tag{3.13}$$

where $\mathcal{M} = \tilde{A}^\top P + P\tilde{A} + \sigma_1 P + (\sigma_2^{-1} + \sigma_3^{-1})P^2 + \sigma_2 g_A^2 I$. It can be verified by using the Schur complement lemma that condition (3.10) holds if and only if the matrix $\mathcal{M}$ is negative definite, say $\mathcal{M} < -\sigma_4 I$ for some $\sigma_4 > 0$. Thus, it follows from (3.13) that

$$V'(t) + \sigma_1 V(t) \leq \sigma_3 h_A^2\|w(t)\|^2,\ t \geq 0,$$

which yields

$$\begin{aligned}V(t) &\leq e^{-\sigma_1 t}\left(V(0) + \sigma_3 h_A^2\int_0^t e^{\sigma_1 s}\|w(s)\|^2 ds\right)\\ &\leq e^{-\sigma_1 t}V(0) + \frac{\sigma_3 h_A^2}{\sigma_1}\|w\|_{L_\infty}^2\left(1 - e^{-\sigma_1 t}\right).\end{aligned}$$

It can be deduced from the last inequality that

$$\limsup_{t\to\infty} V(t) \leq \frac{\sigma_3 h_A^2}{\sigma_1}\|w\|_{L_\infty}^2. \tag{3.14}$$

In addition, according to (3.5) and (3.7), we have

$$\begin{aligned}\|x(t)\| &= \left\|(I - Z_\epsilon(t))^{-1}z(t)\right\|\\ &\leq \frac{1}{1-\sigma}\|z(t)\| = \frac{2}{2 - \epsilon T\eta_A}\|z(t)\|.\end{aligned}$$

This, together with the fact $V(t) \geq \lambda_{\min}(P)\|z(t)\|^2$, leads to

$$\limsup_{t\to\infty}\|x(t)\| \leq \frac{2h_A}{2 - \epsilon T\eta_A}\sqrt{\frac{\sigma_3}{\sigma_1\lambda_{\min}(P)}}\|w\|_{L_\infty},$$

which validates the inequality in (3.11). The proof is completed. □

Remark 3.1. *It can be deduced from the fact that, for any positive scalars σ_3, μ, we have*

$$\left(\frac{1}{\sqrt{\sigma_3}}P - \mu\sqrt{\sigma_3}I\right)^\top\left(\frac{1}{\sqrt{\sigma_3}}P - \mu\sqrt{\sigma_3}I\right) \geq 0,$$

which yields

$$\frac{1}{\sigma_3}P^2 \geq 2\mu P - \mu^2\sigma_3 I.$$

In addition,

$$\sigma_2^{-1}P^2 + \sigma_2 g_A^2 I \geq 2g_A P.$$

Therefore, condition (3.10) *is solvable for a matrix* P *and positive scalars* σ_k $(k = 1, 2, 3)$ *if and only if*

$$\tilde{A}^\top P + P\tilde{A} + (\sigma_1 + 2\mu + 2g_A)P - \mu^2\sigma_3 I < 0. \tag{3.15}$$

For a fixed $\sigma_3 > 0$*, there exist* $\sigma_1 > 0$*,* $\mu > 0$ *satisfying* (3.15) *if and only if*

$$\tilde{A}^\top P + P\tilde{A} + 2g_A P < 0. \tag{3.16}$$

Based on (3.16)*, we have the following result.*

Corollary 3.1. *Let Assumptions* (A1) *and* (A2) *hold. Assume that, for given constants* ϵ*,* T*,* δ_A*,* η_A*, the matrix* $\mathcal{A} = \tilde{A} + g_A I$ *is Hurwitz. Then, the rapidly time-varying system* (3.1) *is ultimately bounded.*

Remark 3.2. *Let* $\epsilon \to 0$*, then* g_A *tends to* δ_A*. Thus, if the matrix* $\tilde{A} + \delta_A I$ *is Hurwitz, then there exist a sufficiently small* $\bar{\epsilon} > 0$ *and a matrix* $P > 0$ *such that condition* (3.16) *holds for any* $\epsilon \in (0, \bar{\epsilon})$*. Based on this observation, we have the following result.*

Corollary 3.2. *Given constants* T*,* δ_A *and* η_A *subject to Assumption* (A1)*. Assume that there exists a matrix* $P > 0$ *satisfying the following Lyapunov inequality*

$$\tilde{A}^\top P + P\tilde{A} + 2\delta_A P < 0. \tag{3.17}$$

Then, system (3.1) *is ultimately bounded for time-scale parameter* $0 < \epsilon < \min(\bar{\epsilon}, 2/T\eta_A)$.

Remark 3.3. *If* $\tilde{A}$ *is a Hurwitz matrix, then*

$$P_0 = \int_0^\infty e^{\tilde{A}^\top t} e^{\tilde{A}t} dt = \int_0^\infty e^{\mathtt{sym}(\tilde{A})t} dt \in \mathbb{R}^{n\times n}$$

is a symmetric positive definite matrix (see, [9], Chapter 2), where $\mathtt{sym}(.)$ *stands for the symmetry operator. Furthermore, we have*

$$\tilde{A}^\top P_0 + P_0\tilde{A} = -I.$$

Thus, condition (3.17) *is clearly satisfied with* $0 < \delta_A < \frac{1}{2\lambda_{\max}(P_0)}$*. On basis of this fact, we have the following explicit boundedness criterion.*

Corollary 3.3. *Let Assumption* (A1) *hold and assume that the matrix* $\tilde{A}$ *is Hurwitz. Then, for given constants* T*,* η_A*, and* δ_A *such that* $0 < \delta_A < \frac{1}{2\lambda_{\max}(P_0)}$*, where* $P_0 = \int_0^\infty e^{\mathtt{sym}(\tilde{A})t} dt$*, there exists an* $\epsilon^* \in (0, 2/T\eta_A)$ *such that the system* (3.1) *is ultimately bounded for any time-scale parameter* $\epsilon \in (0, \epsilon^*)$.

3.2. An application to fast switching systems

In this section, we give a numerical example to illustrate the effectiveness of the result obtained in Theorem 3.1. Let $T > 0, \epsilon > 0$ and consider the following fast switching system

$$x'(t) = A_{\alpha(t/\epsilon)}x(t) + w(t), \tag{3.18}$$

where the switching signal $\alpha : \mathbb{R}_+ \to\in \{1, 2\}$ defined by

$$\alpha(t) = \begin{cases} 1 \text{ if } t \in [kT, (k + 1/2)T), k = 0, 1, 2, \ldots \\ 2 \text{ if } t \in [(k + 1/2)T, (k + 1)T), \end{cases}$$

and the system matrices

$$A_1 = \begin{pmatrix} 0.1 & 0.25 \\ 0.4 & -0.3 \end{pmatrix}, \quad A_2 = \begin{pmatrix} -0.16 & -0.2 \\ -0.35 & 0.1 \end{pmatrix}.$$

Since $\cup_{k\geq 0}[kT, (k + 1)T) = [0, \infty)$, it can be verified that the matrix $A(t) = A_{\alpha(t)}$ can be represented as

$$A(t) = \mathbf{1}_{[kT,(k+1/2)T)}(t)A_1 + \left(1 - \mathbf{1}_{[kT,(k+1/2)T)}(t)\right) A_2,$$

where $\mathbf{1}_{[a,b)}(.)$ is the indicator function of the interval $[a, b)$. It is clear that $A(t)$ is T-periodic and Assumption (A1) is satisfied with $\Delta A(t) = 0$ and

$$\tilde{A} = 0.5A_1 + 0.5A_2 = \begin{pmatrix} -0.03 & 0.025 \\ 0.025 & -0.1 \end{pmatrix}.$$

Note that both the matrices A_1 and A_2 are unstable (not Hurwitz). In addition, we have $\|\tilde{A}\| \simeq 0.108$ and $\eta_A = \max(\|A_1\|, \|A_2\|) \simeq 0.5066$.

For $T = 1$ and $\epsilon = 0.03$, it can be found using the Matlab LMI Toolbox that condition (3.11) is satisfied with $\sigma_1 = 0.001$, $\sigma_2 = 17.0112$, $\sigma_3 = 17.0693$, and

$$P = \begin{pmatrix} 0.3769 & -0.2627 \\ -0.2627 & 1.1124 \end{pmatrix}.$$

By Theorem 3.1, the fast switching system (3.18) is ultimately stable. Moreover, any solution of (3.18) satisfies

$$\limsup_{t\to\infty} \|x(t, x_0, w, \epsilon)\| \leq 71.7715\|w\|_{L_\infty}.$$

On the other hand, for $T = 1$, it is found that condition (3.16) is feasible with $\epsilon_{\max} = 0.5574$. Thus, system (3.18) is ultimately stable for any $\epsilon \in (0, \epsilon_{\max})$.

4. Conclusions

This paper addresses the problem of state bounding for rapidly time-varying linear systems with bounded disturbances. Based on an averaging method and Lyapunov scheme, we derive an explicit bound on the time-scale parameter for which an explicit form of the asymptotic bound of solutions is obtained. An application in fast-switching systems is presented to illustrate the effectiveness of the analysis results.

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