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BIVARIATE POLYNOMIAL INTERPOLATION BASED ON LINE INTEGRALS

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Abstract. We study bivariate polynomial interpolation based on line integrals over line segments connecting two points on two fixed straight lines in the plane. We provide a characterization of the sets of segments that uniquely determine the interpolation polynomial. We also construct illustrative examples for specific cases.
Keywords: Polynomial interpolation, Mean-value interpolation, Line integral.

1. Introduction

Let $\mathcal{P}_d(\mathbb{R}^2)$ be the vector space of polynomials of degree at most d in $\mathbb{R}^2$,

$$\mathcal{P}_d(\mathbb{R}^2) = \text{span}_{\mathbb{R}}\{x^m y^n : 0 \leq m + n \leq d\}.$$

We also consider the space of polynomials

$$\mathcal{Q}_d(\mathbb{R}^2) = \text{span}_{\mathbb{R}}\{x^m y^n : 0 \leq n \leq m \leq d\}.$$

The dimensions of $\mathcal{P}_d(\mathbb{R}^2)$ and $\mathcal{Q}_d(\mathbb{R}^2)$ are both equal to $N_d := (d+1)(d+2)/2$.

A subset $A = \{\mathbf{x}_1, \ldots, \mathbf{x}_{N_d}\}$ of $\mathbb{R}^2$ that consists of N_d distinct points is said to be unisolvent for $\mathcal{P}_d(\mathbb{R}^2)$ if, for every function f defined on A, there exists a unique $P \in \mathcal{P}_d(\mathbb{R}^2)$ such that $f(\mathbf{x}_k) = P(\mathbf{x}_k)$ for $k = 1, \ldots, N_d$. This polynomial is called the Lagrange interpolation polynomial of f at A and is denoted by $\mathbf{L}[A; f]$. Unlike the univariate Lagrange interpolation, the bivariate Lagrange interpolation is not always unisolvent. Moreover, it is difficult to check whether a particular set of N_d distinct points in $\mathbb{R}^2$ is unisolvent. In the literature, many types of unisolvent sets were constructed (see [1]-[4])

In some practical problems, we have information about a function coming as a set of functionals instead of point evaluations. For example, in tomography, the data consists of values of line integrals over segments. These values are called Radon projections. More precisely, let I be a line segment in the plane and $f \in L^1(I)$. The Radon projection $\mathcal{R}(I; f)$ is the line integral of f over I:

$$\mathcal{R}(I; f) = \int\limits_I f(x, y)ds. \tag{1.1}$$

A fundamental problem in this context is the determination of a polynomial from a finite set of its Radon projections. Interpolation theorems serve as a basis for the approximate reconstruction of functions from such projections. Due to the significance of these reconstruction methods in various applications, they have been the subject of extensive study by numerous researchers (see [5]-[7])

We now state an interpolation problem based on Radon projections.

Problem 1. *Let $\mathcal{F}_d$ be the space $\mathcal{P}_d(\mathbb{R}^2)$ or $\mathcal{Q}_d(\mathbb{R}^2)$. Determine a set of line segments $\mathcal{I} = \{I_k : k = 1, \ldots, N_d\}$ such that, for arbitrary real numbers $\gamma_1, \ldots, \gamma_{N_d}$, there exists a unique polynomial $P \in \mathcal{F}_d$ such that*

$$\mathcal{R}(I_k; P) = \gamma_k, \quad k = 1, \ldots, N_d.$$

We say $\mathcal{I}$ regular is $\mathcal{F}_d$ if it solves the problem.

Let $\mathcal{I} = \{I_k : k = 1, \ldots, N_d\}$ be regular for $\mathcal{F}_d$ and let $f \in \bigcap_{k=1}^{N_d} L^1(I_k)$. Let $\mathbf{R}[\mathcal{F}_d, \mathcal{I}; f]$ be the unique polynomial in $\mathcal{F}_d$ such that

$$\mathcal{R}\big(I_k; \mathbf{R}[\mathcal{F}_d, \mathcal{I}; f]\big) = \mathcal{R}\big(I_k; f\big), \quad k = 1, \ldots, N_d. \tag{1.2}$$

The polynomial $\mathbf{R}[\mathcal{F}_d, \mathcal{I}; f]$ is a type of mean-value interpolation polynomial of f.

A natural approach involves selecting line segments that correspond to chords of the unit circle and $\mathcal{F}_d = \mathcal{P}_d(\mathbb{R}^2)$. In [8], the authors constructed a regular set of chords partitioned into $d+1$ groups, with the k-th group consisting of k parallel chords. Bojanov and Xu in [9] demonstrated that a collection of N_d Radon projections, taken over $2[d/2]+1$ parallel chords in each of the $2[(d+1)/2]+1$ equidistant directions, forms a regular set provided that certain matrices, determined by the distances from the origin to the chords, are all non-singular.

A generalization of Problem 1 was studied in [10], where we considered the interpolation problem in $\mathbb{R}^n$. Here, the interpolation conditions are surface integrals over the convex hull of the intersection points of a hyperplane with the coordinate axes. We established a characterization of the hyperplanes such that the interpolation problem has a unique solution. In particular, for $n = 2$, the interpolation conditions are the line integrals over line segments connecting two points on the horizontal and vertical axes. More precisely, we showed that the set of segments connecting $(a_i, 0)$ and $(0, b_i)$, with $a_i b_i \neq 0$, $1 \leq i \leq N_d$, is regular for $\mathcal{P}_d(\mathbb{R}^2)$ if and only if the set of points $\{(a_i, b_i) : 1 \leq i \leq N_d\}$ is unisolvent for $\mathcal{P}_d(\mathbb{R}^2)$.

In this paper, we are interested in characterizing the regular sets of line segments where the endpoints lie on two straight lines in $\mathbb{R}^2$. We consider two distinct cases: intersecting lines and parallel lines. Since the case of two perpendicular lines was treated in [10] (see the above discussion), we only focus on two non-perpendicular lines. Here, the line segment I_k is determined by two parameters a_k and b_k, which form a point $(a_k, b_k) \in \mathbb{R}^2$. In case of two intersecting lines, the main result of Section 2 shows that $\mathcal{I}$ is regular for $\mathcal{P}_d(\mathbb{R}^2)$ if and only if $A = \{(a_k, b_k) : 1 \le k \le N_d\}$ is unisolvent for $\mathcal{P}_d(\mathbb{R}^2)$. Next, we treat the case of parallel lines. We prove in Section 3 that $\mathcal{I}$ is regular for $\mathcal{Q}_d(\mathbb{R}^2)$ if and only if $A = \{(a_k, b_k) : 1 \le k \le N_d\}$ is unisolvent for $\mathcal{P}_d(\mathbb{R}^2)$. We also provide an example in Section 3 showing that the assertion no longer holds when $\mathcal{Q}_d(\mathbb{R}^2)$ is replaced by $\mathcal{P}_d(\mathbb{R}^2)$. Hence, it can be said that the results obtained in this paper complete the theory of polynomial interpolation based on line integrals joining two points on two fixed straight lines in the plane. We also establish relations between the interpolation polynomial induced from $\mathcal{I}$ and the Lagrange interpolation polynomial at A.

Notations and conventions. The set of all nonnegative integers (resp. positive integers) is denoted by $\mathbb{N}$ (resp. $\mathbb{Z}^+$). Throughout the paper, we always assume that d is a positive integer and i, j, k, m, n are natural numbers. We denote by $(x)_k$ the Pochhammer symbol defined by $(x)_k = x(x+1)\cdots(x-k+1)$ for $k \ge 1$ and $(x)_0 = 1$. The monomial $x^m y^n$ with $m, n \in \mathbb{N}$ is denoted by $p_{m,n}(x, y)$.

2. Regular interpolation schemes corresponding to two intersecting lines

We first consider line segments with endpoints lying on two intersecting lines which are not perpendicular. Without loss of generality, we may assume that the two lines are given by $y = 0$ and $y = \alpha x$ with $\alpha \ne 0$. Let $I_\alpha(a; b)$ be the line segment joining $(a, 0)$ and $(b, \alpha b)$ with $(a, b) \ne (0, 0)$ (see Figure 1).

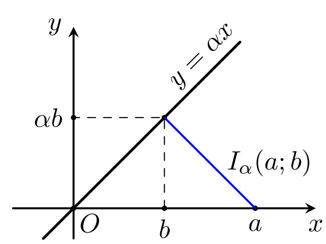

Figure 1. The line segment $I_\alpha(a; b)$

It is parameterized by $x = a + (b - a)t, \quad y = \alpha b t, \quad 0 \le t \le 1.$

Lemma 2.1. *If $p_{m,n}(x,y) = x^m y^n$ with $m, n \in \mathbb{N}$, then*

$$\mathcal{R}(I_\alpha(a;b); p_{m,n}) = \alpha^n \sqrt{(b-a)^2 + (\alpha b)^2} \sum_{k=0}^{m} \frac{(-1)^k (m-k+1)_k b^{m+n-k} (b-a)^k}{(n+1)_{k+1}},$$

where $(x)_k$ is the Pochhammer symbol.

Proof. Using the above paramaterization of $I_\alpha(a;b)$ we have

$$\begin{aligned} \mathcal{R}(I_\alpha(a;b); p_{m,n}) &= \int_0^1 p_{m,n}\big(a + (b-a)t, \alpha b t\big) \sqrt{(b-a)^2 + (\alpha b)^2} dt \\ &= \int_0^1 \big(a + (b-a)t\big)^m \big(\alpha b t\big)^n \sqrt{(b-a)^2 + (\alpha b)^2} dt \\ &= \alpha^n b^n \sqrt{(b-a)^2 + (\alpha b)^2} \int_0^1 t^n \big(a + (b-a)t\big)^m dt. \end{aligned}$$

We need to calculate the last integral. Let us set

$$u(n,m) = \int_0^1 t^n \big(a + (b-a)t\big)^m dt.$$

The integration by parts enables us to write the following recurrence relation

$$\begin{aligned} u(n,m) &= \frac{b^m}{n+1} - \frac{m(b-a)}{n+1} \int_0^1 t^{n+1} \big(a + (b-a)t\big)^{m-1} dt \\ &= \frac{b^m}{n+1} - \frac{m(b-a)}{n+1} u(n+1, m-1). \end{aligned}$$

Using the above relation repeatedly we obtain

$$u(n,m) = \sum_{k=0}^{m} \frac{(-1)^k (m-k+1)_k b^{m-k} (b-a)^k}{(n+1)_{k+1}}.$$

Consequently,

$$\mathcal{R}(I_\alpha(a;b); p_{m,n}) = \alpha^n \sqrt{(b-a)^2 + (\alpha b)^2} \sum_{k=0}^{m} \frac{(-1)^k (m-k+1)_k b^{m+n-k} (b-a)^k}{(n+1)_{k+1}}.$$

The proof is completed. □

Lemma 2.2. *Let d be a positive integer. Then the set of homogeneous polynomials*

$$\Big\{\sum_{k=0}^{m}\frac{(-1)^k(m-k+1)_k y^{d-k}(y-x)^k}{(d-m+1)_{k+1}} : 0\le m\le d\Big\} \tag{2.1}$$

forms a basis for $\mathcal{H}_d(\mathbb{R}^2)$, the space of homogeneous polynomials of degree d in $\mathbb{R}^2$.

Proof. We see that the set $\{y^{d-k}(y-x)^k : 0\le k\le d\}$ forms a basis for $\mathcal{H}_d(\mathbb{R}^2)$. The desired assertion follows directly from the fact that the matrix of coefficients corresponding to the set of polynomials in (2.1) and the above basis has non-zero determinant,

$$\begin{bmatrix} \frac{1}{d+1} & 0 & 0 & 0 & \cdots & 0 \\ \frac{1}{d} & \frac{-1}{d(d+1)} & 0 & 0 & \cdots & 0 \\ \frac{1}{d-1} & \frac{2}{(d-1)d} & \frac{1\cdot 2}{(d-1)d(d+1)} & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{1}{1} & \frac{-d}{2!} & \frac{(d-1)d}{3!} & \frac{-(d-2)(d-1)d}{4!} & \cdots & \frac{(-1)^d d!}{(d+1)!} \end{bmatrix}.$$

The proof is completed. □

Utilizing the above lemma, we immediately obtain the following result.

Lemma 2.3. *Let d be a positive integer. Then, the set of bivariate polynomials*

$$\Big\{q_{m,n}(x,y) := \sum_{k=0}^{m}\frac{(-1)^k(m-k+1)_k y^{m+n-k}(y-x)^k}{(n+1)_{k+1}} : m+n\le d\Big\}$$

forms a basis for $\mathcal{P}_d(\mathbb{R}^2)$.

Theorem 2.1. *Let $\alpha\neq 0$. Then the set of segments $\mathcal{I}=\{I_\alpha(a_i;b_i) : (a_i,b_i)\neq(0,0),\ 1\le i\le N_d\}$ is regular for $\mathcal{P}_d(\mathbb{R}^2)$ if and only if the set of points $A=\{(a_i,b_i) : 1\le i\le N_d\}$ is unisolvent for $\mathcal{P}_d(\mathbb{R}^2)$.*

Proof. Using Lemma 2.1 we get

$$\mathcal{R}(I_\alpha(a;b);p_{m,n}) = \alpha^n\sqrt{(b-a)^2+(\alpha b)^2}\,q_{m,n}(a,b). \tag{2.2}$$

We first assume that A is unisolvent for $\mathcal{P}_d(\mathbb{R}^2)$. Let $P\in\mathcal{P}_d(\mathbb{R}^2)$ such that

$$\mathcal{R}(I_\alpha(a_i;b_i);P)=0,\quad \forall 1\le i\le N_d. \tag{2.3}$$

We need to show that $P = 0$. We write $P(x,y) = \sum_{m+n\le d} c_{m,n} p_{m,n}(x,y)$. From (2.2) we can write

$$\mathcal{R}(I_\alpha(a_i;b_i);P) = \sqrt{(b_i-a_i)^2+(\alpha b_i)^2} \sum_{m+n\le d} \alpha^n c_{m,n} q_{m,n}(a_i,b_i).$$

We define $\widetilde{P}(x,y) = \sum_{m+n\le d} \alpha^n c_{m,n} q_{m,n}(x,y)$. Then $\widetilde{P} \in \mathcal{P}_d(\mathbb{R}^2)$ and

$$\mathcal{R}(I_\alpha(a_i;b_i);P) = \sqrt{(b_i-a_i)^2+(\alpha b_i)^2}\,\widetilde{P}(a_i,b_i), \quad 1 \le i \le N_d. \tag{2.4}$$

Combining (2.3) and (2.4) we obtain $\widetilde{P}(a_i,b_i) = 0$ for $1 \le i \le N_d$. The hypothesis that A is unisolvent gives $\widetilde{P} = 0$. Since $\{q_{m,n} : m+n \le d\}$ is a basis for $\mathcal{P}_d(\mathbb{R}^2)$, we have $c_{m,n} = 0$ for $m+n \le d$. Hence $P = 0$.

Conversely, assume that $\mathcal{I}$ is regular. Let $Q \in \mathcal{P}_d(\mathbb{R}^2)$ such that $Q(a_i,b_i) = 0$ for $1 \le i \le N_d$. We write $Q = \sum_{m+n\le d} d_{m,n} q_{m,n}$ and define $\widehat{Q} = \sum_{m+n\le d} \frac{d_{m,n}}{\alpha^n} p_{m,n}$. By the above arguments we get

$$\mathcal{R}(I_\alpha(a_i;b_i);\widehat{Q}) = \sqrt{(b_i-a_i)^2+(\alpha b_i)^2}\,Q(a_i,b_i), \quad 1 \le i \le N_d.$$

Hence $\mathcal{R}(I_\alpha(a_i;b_i);\widehat{Q}) = 0, \quad 1 \le i \le N_d$. Consequently, $\widehat{Q} = 0$, because $\mathcal{I}$ is regular for $\mathcal{P}_d(\mathbb{R}^2)$. This forces $d_{m,n} = 0$ for $m+n \le d$, and hence $Q = 0$. The proof is completed. $\square$

Proposition 2.1. *Let* $\alpha \ne 0$. *Let* $\mathcal{I} = \{I_\alpha(a_i;b_i) : (a_i,b_i) \ne (0,0),\ 1 \le i \le N_d\}$ *be regular for* $\mathcal{P}_d(\mathbb{R}^2)$ *and* $A = \{(a_i,b_i) : 1 \le i \le N_d\}$. *Let* $f \in \bigcap_{i=1}^{N_d} L^1(I_\alpha(a_i;b_i))$ *and* $f^\star : A \to \mathbb{R}$ *define by*

$$f^\star(a_i,b_i) = \frac{\mathcal{R}\big(I_\alpha(a_i;b_i);f\big)}{\sqrt{(b_i-a_i)^2+(\alpha b_i)^2}}, \quad 1 \le i \le N_d.$$

Let $\mathbf{R}[\mathcal{P}_d(\mathbb{R}^2),\mathcal{I};f] = \sum\limits_{m+n\le d} c_{m,n} p_{m,n}$ *and* $\mathbf{L}[A;f^\star] = \sum\limits_{m+n\le d} c^\star_{m,n} q_{m,n}$.
Then $c^\star_{m,n} = \alpha^n c_{m,n}$ *for* $m+n \le d$.

Proof. Lemma 2.1 gives

$$\begin{aligned}\mathcal{R}\big(I_\alpha(a_i;b_i);f\big) &= \mathcal{R}\big(I_\alpha(a_i;b_i);\mathbf{R}[\mathcal{P}_d(\mathbb{R}^2),\mathcal{I};f]\big) = \sum_{m+n\le d} c_{m,n}\mathcal{R}\big(I_\alpha(a_i;b_i);p_{m,n}\big)\\ &= \sqrt{(b_i-a_i)^2+(\alpha b_i)^2}\sum_{m+n\le d}\alpha^n c_{m,n} q_{m,n}(a_i,b_i).\end{aligned}$$

It follows that

$$\sum_{m+n\le d} \alpha^n c_{m,n} q_{m,n}(a_i,b_i) = f^\star(a_i,b_i), \quad 1 \le i \le N_d, \tag{2.5}$$

The polynomial $\sum_{m+n\le d}\alpha^n c_{m,n}q_{m,n}$ belonging the space $\mathcal{P}_d(\mathbb{R}^2)$ interpolates $f^\star$ at A. By the uniqueness of Lagrange interpolation, we have

$$\mathbf{L}[A;f^\star]=\sum_{m+n\le d}\alpha^n c_{m,n}q_{m,n}.$$

The desired relation follows directly from the last equality. The proof is completed. □

3. Regular interpolation schemes corresponding to two parallel lines

We now consider line segments with endpoints located on two parallel lines. There is no loss of generality in assuming the two lines are $y=0$ and $y=\beta$ where $\beta\neq 0$. Let $J_\beta(a;b)$ denote the line segment connecting the points $(a,0)$ and (b,β) (see Figure 2).

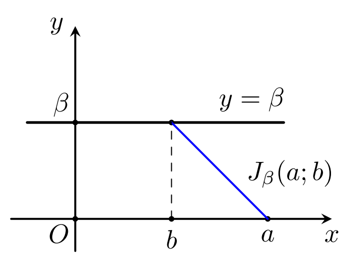

Figure 2. The line segment $J_\beta(a;b)$

This segment can be parameterized by $x=a+(b-a)t,\quad y=\beta t,\quad 0\le t\le 1.$

Lemma 3.1. *If $p_{m,n}(x,y)=x^m y^n$ with $m,n\in\mathbb{N}$, then*

$$\mathcal{R}(J_\beta(a;b);p_{m,n})=\beta^n\sqrt{(b-a)^2+\beta^2}\sum_{k=0}^{m}\frac{(-1)^k(m-k+1)_k b^{m-k}(b-a)^k}{(n+1)_{k+1}},$$

where $(x)_k$ is the Pochhammer symbol.

Proof. We can write

$$\begin{aligned}\mathcal{R}(J_\alpha(a;b);p_{m,n}) &= \int_0^1 p_{m,n}\big(a+(b-a)t,\beta t\big)\sqrt{(b-a)^2+\beta^2}dt\\ &= \int_0^1 \big(a+(b-a)t\big)^m\big(\beta t\big)^n\sqrt{(b-a)^2+\beta^2}dt\\ &= \beta^n\sqrt{(b-a)^2+\beta^2}\int_0^1 t^n\big(a+(b-a)t\big)^m dt.\end{aligned}$$

Using the computation of the integral $\int_0^1 t^n\big(a+(b-a)t\big)^m dt$ in Lemma 2.1, we obtain the desired relation. □

Lemma 3.2. *Let m be a positive integer. Then the set of homogeneous polynomials*

$$\Big\{\sum_{k=0}^{m}\frac{(-1)^k(m-k+1)_k y^{m-k}(y-x)^k}{(n+1)_{k+1}} : 0\le n\le m\Big\} \tag{3.1}$$

forms a basis for $\mathcal{H}_m(\mathbb{R}^2)$, the space of homogeneous polynomials of degree m in $\mathbb{R}^2$.

Proof. As in the proof of Lemma 2.2, since$\{y^{m-k}(y-x)^k : 0\le k\le m\}$ is a basis for $\mathcal{H}_m(\mathbb{R}^2)$, it suffices to show that the matrix of coefficients corresponding to the set of polynomials in (3.1) and the above basis is invertible

$$M=\begin{bmatrix} \frac{1}{1} & \frac{-m}{1\cdot2} & \frac{(m-1)m}{1\cdot2\cdot3} & \cdots & \frac{(-1)^m 1\cdot2\cdots m}{1\cdot2\cdots(m+1)}\\ \frac{1}{2} & \frac{-m}{2\cdot3} & \frac{(m-1)m}{2\cdot3\cdot4} & \cdots & \frac{(-1)^m 1\cdot2\cdots m}{2\cdot3\cdots(m+2)}\\ \frac{1}{3} & \frac{-m}{3\cdot4} & \frac{(m-1)m}{3\cdot4\cdot5} & \cdots & \frac{(-1)^m 1\cdot2\cdots m}{3\cdot4\cdots(m+3)}\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ \frac{1}{m+1} & \frac{-m}{(m+1)(m+2)} & \frac{(m-1)m}{(m+1)(m+2)(m+3)} & \cdots & \frac{(-1)^m 1\cdot2\cdots m}{(m+1)(m+2)\cdots(2m+1)} \end{bmatrix}.$$

It is easily check that $\det M=(-1)^{\frac{m(m+1)}{2}}\prod_{k=0}^m(m-k+1)_k\prod_{k=1}^m k!\det H$, where

$$H:=\begin{bmatrix} \frac{1}{1!} & \frac{1}{2!} & \frac{1}{3!} & \cdots & \frac{1}{(m+1)!}\\ \frac{1}{2!} & \frac{1}{3!} & \frac{1}{4!} & \cdots & \frac{1}{(m+2)!}\\ \frac{1}{3!} & \frac{1}{4!} & \frac{1}{5!} & \cdots & \frac{1}{(m+3)!}\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ \frac{1}{(m+1)!} & \frac{1}{(m+2)!} & \frac{1}{(m+3)!} & \cdots & \frac{1}{(2m+1)!} \end{bmatrix}.$$

We see that H is a Hankel matrix. The determinant of this matrix can be found in the literature. To make it easier for readers to follow, we give a detailed computation. For each $1\le j\le m+1$, we factor out $\frac{1}{(m+j)!}$ from the j-th column and get

$$\det H=\frac{1}{\prod_{j=1}^{m+1}(m+j)!}\det\Big[\frac{(m+j)!}{(k+j-1)!}\Big]_{k,j=1}^{m+1}.$$

Note that $\frac{(m+j)!}{(k+j-1)!}=f_{m+1-k}(m+j)$, where $f_k(x)=x(x-1)\cdots(x-k+1)$ for $k\ge1$ and $f_0(x)=1$. Hence

$$\begin{aligned}\det H &= \frac{1}{\prod_{j=1}^{m+1}(m+j)!}\det\Big[f_{m+1-k}(m+j)\Big]_{k,j=1}^{m+1}\\ &= \frac{(-1)^{\frac{m(m+1)}{2}}}{\prod_{j=1}^{m+1}(m+j)!}\det\Big[f_{k-1}(m+j)\Big]_{k,j=1}^{m+1}.\end{aligned}$$

We will reduce the last determinant to the Vandermonde determinant. Indeed, for any set $\{x_1, \ldots, x_{m+1}\}$ of $m+1$ real numbers, since $f_k(x)$ is a monic polynomial of degree k, we can use the row operation of matrix to obtain

$$\det\Big[f_{k-1}(x_j)\Big]_{k,j=1}^{m+1} = \det\Big[g_{k-1}(x_j)\Big]_{k,j=1}^{m+1} = \prod_{1\le k<j\le m+1}(x_j - x_k),$$

where $g_k(x) = x^k$ for $k = 0, \ldots, m$. It follows that

$$\det\Big[f_{k-1}(m+j)\Big]_{k,j=1}^{m+1} = \prod_{1\le k<j\le m+1}[(m+j)-(m+k)] = \prod_{1\le k<j\le m+1}(j-k) = \prod_{i=1}^{m} i!.$$

Hence $\det H = \frac{(-1)^{\frac{m(m+1)}{2}}\prod_{i=1}^m i!}{\prod_{j=1}^{m+1}(m+j)!}$. Combining the above computations, we obtain $\det M = \frac{\prod_{k=0}^m (m-k+1)_k \prod_{i=1}^m (i!)^2}{\prod_{j=1}^{m+1}(m+j)!}$. Consequently, M is invertible, and the proof is completed. □

Applying the above lemma, we get the following result.

Lemma 3.3. *Let d be a positive integer. Then the set of polynomials*

$$\Big\{r_{m,n}(x,y) := \sum_{k=0}^{m}\frac{(-1)^k(m-k+1)_k y^{m-k}(y-x)^k}{(n+1)_{k+1}} : 0 \le n \le m \le d\Big\} \tag{3.2}$$

forms a basis for $\mathcal{P}_d(\mathbb{R}^2)$.

Using similar arguments presented in the proof of Theorem 2.1 and Proposition 2.1, we get the following results.

Theorem 3.1. *Let $\beta \neq 0$. Then the set of segments $\mathcal{J} = \{J_\beta(a_i; b_i) : 1 \le i \le N_d\}$ is regular for $\mathcal{Q}_d(\mathbb{R}^2)$ if and only if the set of points $A = \{(a_i, b_i) : 1 \le i \le N_d\}$ is unisolvent for $\mathcal{P}_d(\mathbb{R}^2)$.*

Proof. In view of Lemma 3.1 we have

$$\mathcal{R}(J_\beta(a;b); p_{m,n}) = \beta^n\sqrt{(b-a)^2+\beta^2}\, r_{m,n}(a,b). \tag{3.3}$$

We first assume that A is unisolvent for $\mathcal{P}_d(\mathbb{R}^2)$. Let $P \in \mathcal{Q}_d(\mathbb{R}^2)$ satisfying the condition

$$\mathcal{R}(J_\alpha(a_i;b_i); P) = 0, \quad 1 \le i \le N_d. \tag{3.4}$$

We need to show that $P = 0$. Wring $P(x,y) = \sum_{0\le n\le m\le d} c_{m,n}p_{m,n}(x,y)$ and using (3.3) we can write

$$\mathcal{R}(J_\beta(a_i;b_i); P) = \sqrt{(b_i-a_i)^2+\beta^2}\sum_{0\le n\le m\le d}\beta^n c_{m,n} r_{m,n}(a_i,b_i).$$

Let us set $\widetilde{P}(x,y)=\sum_{0\le n\le m\le d}\beta^n c_{m,n}r_{m,n}(x,y)$. Observe that $\widetilde{P}\in\mathcal{P}_d(\mathbb{R}^2)$ and

$$\mathcal{R}(J_\beta(a_i;b_i);P)=\sqrt{(b_i-a_i)^2+\beta^2}\,\widetilde{P}(a_i,b_i),\quad 1\le i\le N_d. \tag{3.5}$$

Combining (3.4) and (3.5) we obtain $\widetilde{P}(a_i,b_i)=0$ for $1\le i\le N_d$. Since A is unisolvent for $\mathcal{P}_d(\mathbb{R}^2)$, we get $\widetilde{P}=0$. It follows that $c_{m,n}=0$ for $0\le n\le m\le d$ because $\{r_{m,n}:0\le n\le m\le d\}$ is a basis for $\mathcal{P}_d(\mathbb{R}^2)$. This forces $P=0$.

Conversely, assume that $\mathcal{J}$ is regular for $\mathcal{Q}_d(\mathbb{R}^2)$. Let $Q\in\mathcal{P}_d(\mathbb{R}^2)$ such that $Q(a_i,b_i)=0,\quad 1\le i\le N_d$. Since the set $\{r_{m,n}:0\le n\le m\le d\}$ form a basis for $\mathcal{P}_d(\mathbb{R}^2)$, we can write $Q=\sum_{0\le n\le m\le d}d_{m,n}r_{m,n}$. Let us define $\widehat{Q}=\sum_{0\le n\le m\le d}\frac{d_{m,n}}{\beta^n}p_{m,n}$. We have $\widehat{Q}\in\mathcal{Q}_n$. By the above arguments, we get

$$\mathcal{R}(J_\beta(a_i;b_i);\widehat{Q})=\sqrt{(b_i-a_i)^2+\beta^2}\,Q(a_i,b_i),\quad 1\le i\le N_d.$$

It follows that $\mathcal{R}(J_\beta(a_i;b_i);\widehat{Q})=0,\quad 1\le i\le N_d$. Consequently, $\widehat{Q}=0$, because $\mathcal{J}$ is regular for $\mathcal{Q}_d(\mathbb{R}^2)$. Hence, $d_{m,n}=0$ for $0\le n\le m\le d$. This forces $Q=0$, and the proof is completed. □

Proposition 3.1. *Let $\beta\neq 0$. Let $\mathcal{J}=\{J_\beta(a_i;b_i):1\le i\le N_d\}$ be regular for $\mathcal{Q}_d(\mathbb{R}^2)$ and $A=\{(a_i,b_i):1\le i\le N_d\}$. Let $f\in\bigcap_{i=1}^{N_d}L^1(J_\beta(a_i;b_i))$ and $f^*:A\to\mathbb{R}$ define by*

$$f^*(a_i,b_i)=\frac{\mathcal{R}\big(J_\beta(a_i;b_i);f\big)}{\sqrt{(b_i-a_i)^2+\beta^2}},\quad 1\le i\le N_d.$$

Let $\mathbf{R}[\mathcal{Q}_d(\mathbb{R}^2),\mathcal{J};f]=\sum\limits_{0\le n\le m\le d}c_{m,n}p_{m,n}$ *and* $\mathbf{L}[A;f^*]=\sum\limits_{0\le n\le m\le d}c^*_{m,n}r_{m,n}$.
Then $c^*_{m,n}=\beta^n c_{m,n}$ for $0\le n\le m\le d$.

Proof. From Lemma 3.1 we can write

$$\begin{aligned}\mathcal{R}\big(J_\beta(a_i;b_i);f\big)&=\mathcal{R}\big(J_\beta(a_i;b_i);\mathbf{R}[\mathcal{Q}_d(\mathbb{R}^2),\mathcal{J};f]\big)=\sum_{0\le n\le m\le d}c_{m,n}\mathcal{R}\big(J_\beta(a_i;b_i);p_{m,n}\big)\\&=\sqrt{(b_i-a_i)^2+\beta^2}\sum_{0\le n\le m\le d}\beta^n c_{m,n}r_{m,n}(a_i,b_i).\end{aligned}$$

It follows that

$$\sum_{0\le n\le m\le d}\beta^n c_{m,n}r_{m,n}(a_i,b_i)=f^*(a_i,b_i),\quad 1\le i\le N_d, \tag{3.6}$$

Observe that $\sum_{0\le n\le m\le d}\beta^n c_{m,n}r_{m,n}$ is a polynomial of degree at most d in $\mathbb{R}^2$. Relation (3.6) induces that this polynomial interpolates f^* at A. By the uniqueness of Lagrange interpolation, we have

$$\mathbf{L}[A;f^*]=\sum_{0\le n\le m\le d}\beta^n c_{m,n}r_{m,n}.$$

Since $\{r_{m,n}:0\le n\le m\le d\}$ forms a basis for $\mathcal{P}_d(\mathbb{R}^2)$, we get $c^*_{m,n}=\beta^n c_{m,n}$ for $0\le n\le m\le d$. □

Remark 3.1. *The space of polynomial $\mathcal{Q}_d(\mathbb{R}^d)$ arises naturally from Lemma 3.1. Therefore, it is reasonable to work with $\mathcal{Q}_d(\mathbb{R}^d)$ instead of $\mathcal{P}_d(\mathbb{R}^d)$. The problem of characterizing the regularity of $\mathcal{J}$ corresponding to $\mathcal{P}_d(\mathbb{R}^d)$ remains open.*

4. Examples

In this section, we compute the interpolation polynomials constructed in the two preceding sections. While Propositions 2.1 and 3.1 provide explicit formulas for these polynomials, our examples focus on the case $d = 1$, which allows for direct computation.

Example 4.1. *We consider three non-collinear points $\mathbf{a} = (1, 1)$, $\mathbf{b} = (2, 1)$ and $\mathbf{c} = (3, 2)$ in the plane. It is well-known that $A = \{\mathbf{a}, \mathbf{b}, \mathbf{c}\}$ is unisolvent for $\mathcal{P}_1(\mathbb{R}^2)$. Let us take $\alpha = 2$. By Theorem 2.1, the set of three line segments*

$$\mathcal{I} = \{I_2(1; 1), I_2(2; 1), I_2(3; 2)\}$$

is regular for $\mathcal{P}_1(\mathbb{R}^2)$.

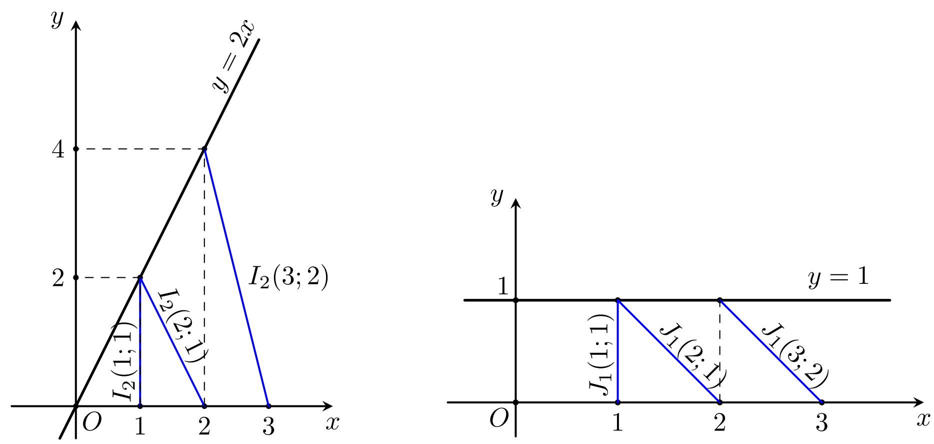

***Figure 3. Three line segments in $\mathcal{I}$* (left) and $\mathcal{J}$ (right)**

Assume that we have the information of a function f given by

$$\mathcal{R}(I_2(1; 1); f) = \gamma_1, \quad \mathcal{R}(I_2(2; 1); f) = \gamma_2, \quad \mathcal{R}(I_2(3; 2); f) = \gamma_3.$$

We need to compute the interpolation polynomial

$$P(x, y) := \mathbf{R}[\mathcal{P}_1(\mathbb{R}^2), \mathcal{I}; f](x, y) = u + vx + wy.$$

Direct computation gives

$$\mathcal{R}(I_2(1; 1); P) = u\mathcal{R}(I_2(1; 1); 1) + v\mathcal{R}(I_2(1; 1); x) + w\mathcal{R}(I_2(1; 1); y) = 2u + 2v + 2w;$$

$$\mathcal{R}(I_2(2; 1); P) = u\mathcal{R}(I_2(2; 1); 1)+v\mathcal{R}(I_2(2; 1); x)+w\mathcal{R}(I_2(2; 1); y) = \sqrt{5}u+\frac{3\sqrt{5}}{2}v+\sqrt{5}w;$$

$$\mathcal{R}(I_2(3;2);P) = u\mathcal{R}(I_2(3;2);1)+v\mathcal{R}(I_2(3;2);x)+w\mathcal{R}(I_2(3;2);y) = \sqrt{17}u+\frac{5\sqrt{17}}{2}v+2\sqrt{17}w.$$

Hence we get a system of equations

$$2u+2v+2w = \gamma_1, \sqrt{5}u+\frac{3\sqrt{5}}{2}v+\sqrt{5}w = \gamma_2, \sqrt{17}u+\frac{5\sqrt{17}}{2}v+2\sqrt{17}w = \gamma_3.$$

It has the root

$$u = \frac{1}{2}\gamma_1+\frac{\sqrt{5}}{5}\gamma_2-\frac{\sqrt{17}}{17}\gamma_3, \quad v = -\gamma_1+\frac{2\sqrt{5}}{5}\gamma_2, \quad w = \gamma_1-\frac{3\sqrt{5}}{5}\gamma_2+\frac{\sqrt{17}}{17}\gamma_3.$$

It follows that

$$P(x,y) = \frac{1}{2}\gamma_1+\frac{\sqrt{5}}{5}\gamma_2-\frac{\sqrt{17}}{17}\gamma_3+\big(-\gamma_1+\frac{2\sqrt{5}}{5}\gamma_2\big)x+\big(\gamma_1-\frac{3\sqrt{5}}{5}\gamma_2+\frac{\sqrt{17}}{17}\big)y.$$

Example 4.2. *Let $A = \{\mathbf{a},\mathbf{b},\mathbf{c}\}$ be the set of points in the previous example. We will show that set of three line segments*

$$\mathcal{J} = \{J_1(1;1), J_1(2;1), J_1(3;2)\}$$

is not regular for $\mathcal{P}_1(\mathbb{R}^2)$. We have

$$\begin{vmatrix} \mathcal{R}(J_1(1;1);1) & \mathcal{R}(J_1(1;1);x) & \mathcal{R}(J_1(1;1);y) \\ \mathcal{R}(J_1(2;1);1) & \mathcal{R}(J_1(2;1);x) & \mathcal{R}(J_1(2;1);y) \\ \mathcal{R}(J_1(3;2);1) & \mathcal{R}(J_1(3;2);x) & \mathcal{R}(J_1(3;2);y) \end{vmatrix} = \begin{vmatrix} 1 & 1 & \frac{1}{2} \\ \sqrt{2} & \frac{3\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ \sqrt{2} & \frac{5\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{vmatrix} = 0.$$

Hence, the coefficient matrix is not invertible. Consequently, $\mathcal{J}$ is not regular for $\mathcal{P}_1(\mathbb{R}^2)$.

Example 4.3. *Let $A = \{\mathbf{a},\mathbf{b},\mathbf{c}\}$ be the set of points in Example 4.1. Let us take $\beta = 1$. Using Theorem 3.1, we get a regular set for $\mathcal{Q}_1(\mathbb{R}^2) = span_{\mathbb{R}}\{1,x,xy\}$ consisting of three line segments*

$$\mathcal{J} = \{J_1(1;1), J_1(2;1), J_1(3;2)\}$$

Assume that we have the information of a function f given by

$$\mathcal{R}(J_1(1;1);f) = \gamma_1, \quad \mathcal{R}(J_1(2;1);f) = \gamma_2, \quad \mathcal{R}(J_1(3;2);f) = \gamma_3.$$

We want to compute the interpolation polynomial

$$Q(x,y) := \mathbf{R}[\mathcal{Q}_1(\mathbb{R}^2),\mathcal{J};f](x,y) = u+vx+wxy.$$

By direct computations, we have

$$\mathcal{R}(J_1(1;1);Q) = u\mathcal{R}(J_1(1;1);1)+v\mathcal{R}(J_1(1;1);x)+w\mathcal{R}(J_1(1;1);xy) = u+v+\frac{1}{2}w;$$

$$\mathcal{R}(J_1(2;1);Q) = u\mathcal{R}(J_1(2;1);1)+v\mathcal{R}(J_1(2;1);x)+w\mathcal{R}(J_1(2;1);xy) = \sqrt{2}u+\frac{3\sqrt{2}}{2}v+\frac{2\sqrt{2}}{3}w;$$

$$\mathcal{R}(J_1(3;2);Q) = u\mathcal{R}(J_1(3;2);1)+v\mathcal{R}(J_1(3;2);xy)+w\mathcal{R}(J_1(3;2);y) = \sqrt{2}u+\frac{5\sqrt{2}}{2}v+\frac{7\sqrt{2}}{6}w.$$

Hence, we get a system of equations

$$u + v + \frac{1}{2}w = \gamma_1, \sqrt{2}u + \frac{3\sqrt{2}}{2}v + \frac{2\sqrt{2}}{3}w = \gamma_2, \sqrt{2}u + \frac{5\sqrt{2}}{2}v + \frac{7\sqrt{2}}{6}w = \gamma_3.$$

Its root can be computed easily,

$$u = \gamma_1 + \frac{\sqrt{2}}{2}\gamma_2 - \frac{\sqrt{2}}{2}\gamma_3, \quad v = -6\gamma_1 + 4\sqrt{2}\gamma_2 - \sqrt{2}\gamma_3, \quad w = 12\gamma_1 - 9\sqrt{2}\gamma_2 + 3\sqrt{2}\gamma_3.$$

Consequently,

$$Q(x,y) = \gamma_1 + \frac{\sqrt{2}}{2}\gamma_2 - \frac{\sqrt{2}}{2}\gamma_3 + (-6\gamma_1 + 4\sqrt{2}\gamma_2 - \sqrt{2}\gamma_3)x + (12\gamma_1 - 9\sqrt{2}\gamma_2 + 3\sqrt{2}\gamma_3)xy.$$

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