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USING GEOGEBRA FOR TEACHING PROBLEMS ON FIXED POINTS OF FAMILIES OF LINES AND CIRCLES IN SECONDARY SCHOOL GEOMETRY

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Abstract. This paper explores the application of GeoGebra software to assist teachers and students in lessons on fixed points of families of lines and circles, a relatively complex topic in secondary school geometry that requires logical reasoning and advanced proof skills. GeoGebra, as a visual modeling tool, enables the discovery and proof of fixed points in problems involving families of lines and circles. The selected illustrations are derived from problems featured in competitions for high-achieving students and specialized 10th-grade entrance exams in recent years. To evaluate the effectiveness of this teaching method, experiments were conducted with 59 Grade 9 students at Newton Secondary and High School (Bac Tu Liem, Ha Noi). A t-test on students' test scores after being taught fixed-point problems showed statistically significant differences between the GeoGebra method group ($M = 7.07$, $SD = 1.64$) and the traditional method group ($M = 5.6$, $SD = 1.3$) [$t(59) = 1.987934$, $p = 0.001134 < 0.05$]. Additionally, students in the experimental group demonstrated greater interest in the presented problems. The findings from this study highlight the potential of using GeoGebra to teach advanced geometrical problems within the Vietnamese educational context.

Keywords: mathematics education, Geometry software, dynamic geometry.

1. Introduction

Families of lines or circles passing through a fixed point are a common type of advanced problem for secondary school students. This type of problem is both interesting and challenging, often posing difficulties for students in finding solutions [1].

In this article, the use of GeoGebra to find fixed points is illustrated. GeoGebra, a free software application available for both PCs and smartphones, is widely used in teaching mathematics. With support for the Vietnamese language, it is effective for teaching at both elementary and college levels. Its integration of dynamic geometry and analytical calculation makes it particularly convenient for combining geometry and algebra [2]. The use of GeoGebra as an interactive and visual tool in teaching mathematics has been explored in several previous studies by the authors [3]-[10]. In this context, GeoGebra has the potential to serve as an effective tool for providing visualizations that help students discover solutions to problems involving fixed points.

The following section demonstrates how GeoGebra can be used in classroom settings with promising experimental results. To find the fixed point, the problem can first be visualized in GeoGebra. The point or line can then be moved to observe the common point of a family of lines or circles. Alternatively, only three cases can be drawn to find the intersection of the lines or circles. Based on this visualization, students can predict the properties of the fixed points and attempt to develop a mathematical proof.

2. Content

2.1. Application of GeoGebra to problems involving fixed points

The process of teaching fixed-point problems using GeoGebra will be described in detail for the first problem. For subsequent problems, the procedures are similar; therefore, only the GeoGebra solutions and proofs are provided.

***Problem 1**:* Given a circle with center O and a fixed chord AB. C is the point moving on the small arc AB. Let M be the midpoint of BC. From M, draw MN perpendicular to AC ($N \in AC$). Prove that the line MN always passes through a fixed point.

***Solution**:*

• *Step 1*: Use Geogebra for setting up the lines or circles that move

Draw the circle with center O. Pick two points A and B on the circle. Draw the chord AB. Draw diameter AD. Let C be a point on the arc AB. Draw line AC and use the tool inside GeoGebra to find the midpoint M. Use the perpendicular line tool to draw MN perpendicular to AC.

• *Step 2:* Use Geogebra to visualize the movement and predict the fixed point(s) when C moves along AB. Prediction: MN always passes through the midpoint of BD.

• *Step 3*: Instructions for proof: Teachers guide students to the solution with questions

The teacher asks students to extend MN to intersect BD at E and to prove that E is the midpoint of BD.

-Teacher: Observe and comment on the relative positions between MN and CD when C moves along the smaller arc AB.

-Student: $MN//CD$ (because $\widehat{ACD} = \widehat{ANM} = 90^0$).

Hence, students can use Thales's theorem to conclude that E is the midpoint BD.

• *Step 4***:** Verification: The teacher asks students to verify with tools in GeoGebra.

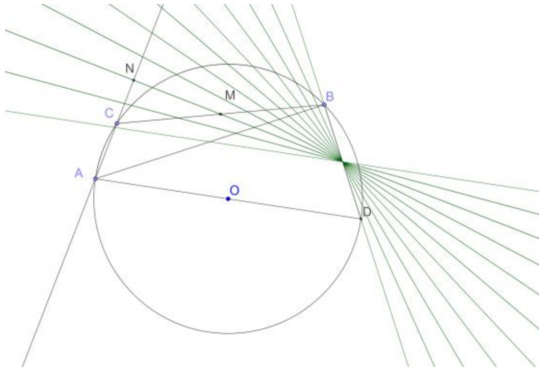

Figure 1. Fixed point of a family of lines for problem 2 in GeoGebra

Proof. Angles $\widehat{ACD} = \widehat{ANM} = 90^0$. Hence, $MN//CD$. Since M is the midpoint of BC, according to Thales' Theorem, E is the midpoint of BD. As such, MN always passes through E.

Problem 2: Given circle (C) with center O and a point A inside (C) but not congruent with O. A moving circle goes through A but not O intersects (C) at M and N. Prove that the circumcircle of triangle OMN always passes through a fixed point different from O.

Solution: Draw a circle with center O. Select a point A inside the circle. Pick point M on the circle. Line AM intersects (O) at N. Draw the circumcircle of triangle OMN using the tool inside GeoGebra. Line AO intersects the circle (O) at C and D; and the circle (OMN) at B. Moving M along the circle (O), the circle (OMN) always passes through a fixed point B (Figure 2).

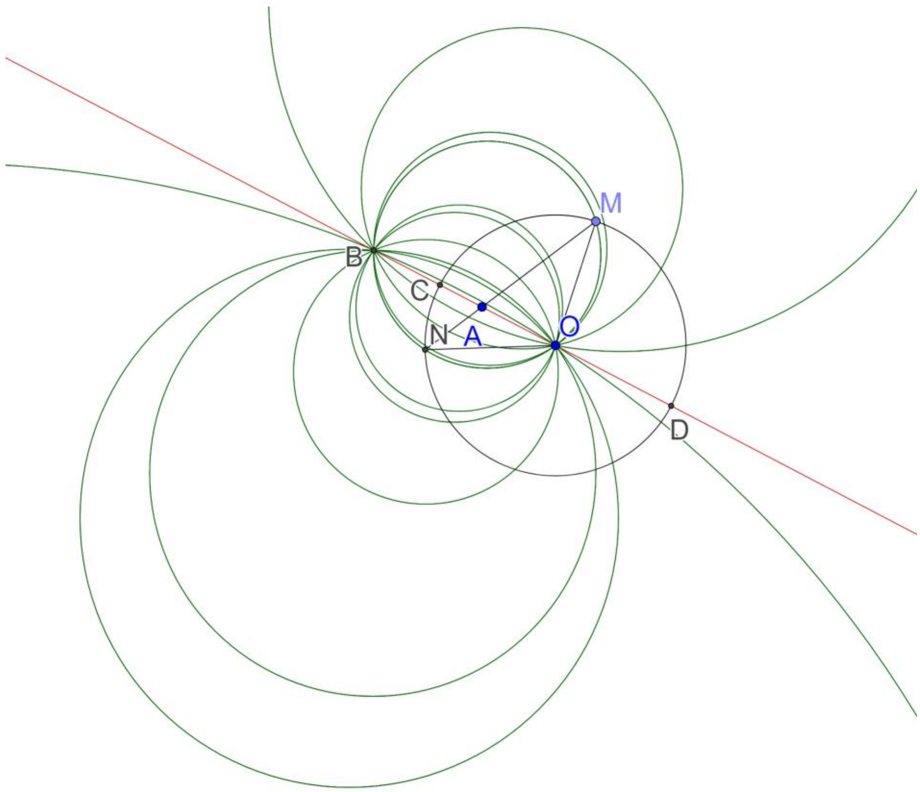

Figure 2. Fixed point of a family of circles for problem 2 in GeoGebra

Proof. Since A and O are fixed, C and D are also fixed points. $\Delta CAN \sim \Delta MAD$ (angle-angle). Hence, $\frac{CA}{MA} = \frac{AN}{AD}$, or $CA \cdot DA = AN \cdot MA$. Similarly, $\Delta BMA \sim \Delta NOA$ (angle-angle). Hence $\frac{AB}{AM} = \frac{AN}{AO}$ or $AB \cdot AO = AN \cdot AM$. Consequently $AB \cdot AO = AC \cdot AD$ or $AB = \frac{AC \times AD}{AO}$ Is constant. Meanwhile, B is on the fixed line AO so B is also a fixed point.

Problem 3: Given circle (C) with center O and a line d outside of the circle. Point I moves along d. The circle of diameter OI intersects (C) at M and N. This Proves that line MN always passes through a fixed point.

Solution: Draw a circle with center O and line d. Select point I on d. Draw a circle with diameter OI, which intersects (C) at M and N. Draw OA perpendicular to d, intersecting MN at B. As I move along d, MN always passes through the fixed point B.

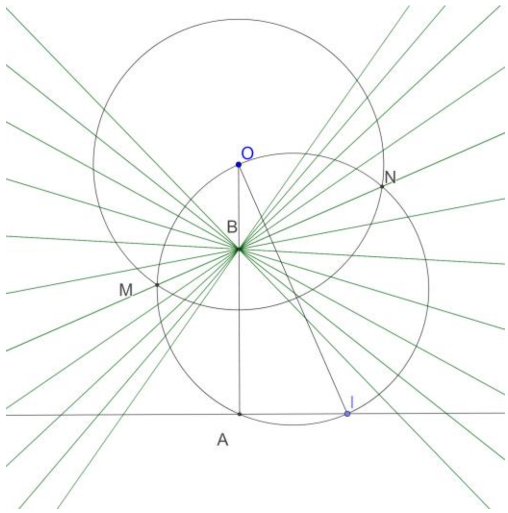

Figure 3. Fixed point of a family of the line for problem 3 in GeoGebra

Proof. Triangles OCB and OAI are similar. Hence $\frac{OC}{OB} = \frac{OA}{OI}$ or $OC \cdot OI = OA \cdot OB$. Triangle IMO is a right triangle. Hence $OC \cdot OI = OM^2 = R^2$. So $OA \cdot OB = R^2$ so OB is of constant length. Point B is on the perpendicular line to d from O so B is a fixed point.

***Problem 4**:* Given half circle with diameter AB. C is a point on this half-circle. On the outside of the triangles, ABC draws the squares BCDE and ACFG. Let Ax and By be the tangents of the half circles. Prove that when C moves along the half circle, the line ED always passes through a fixed point. Do the same for line FG.

***Solution**:* Draw the half circle with Diameter AB and the tangents Ax, By perpendicular to AB. Let C be a point on arc AB. Construct squares CBED and CAGF using the tool for regular polygon in GeoGebra. DE intersects By at I, GF intersects Ax at K. I and K are fixed points.

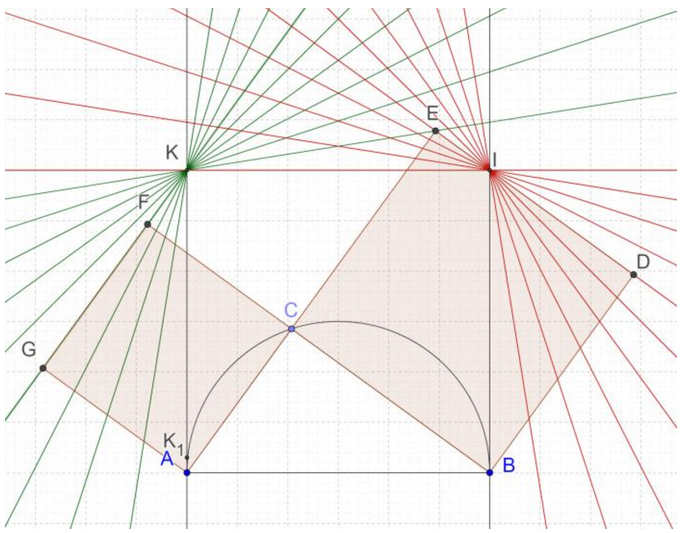

Figure 4. Fixed point of a family of the line for problem 4 in GeoGebra

Proof. Right triangles ABC and BIE are congruent because $\widehat{IBE} = \widehat{ABC}$ ($\widehat{CBI} = 90^0$) and BC=BE. Hence BI=AB. So it is a fixed point. The same can be proven for K in a similar way. K and I are also two vertices of the square of side AB.

Problem 5: Given the isosceles triangle ABC (AB =AC). D and E are points on AB and BC, respectively, so that the projection of DE on BC is half of BC. This proves that the line perpendicular to DE always passes through a fixed point.

Solution: Draw segment BC and its perpendicular bisector. Select point A on this bisector. Connect AB and AC. Select point D on AB. E is the intersection of the circle $(F, BC/2)$ and BC. AD moves along AB, the line perpendicular to DE always passes through a fixed point on the perpendicular bisector of BC.

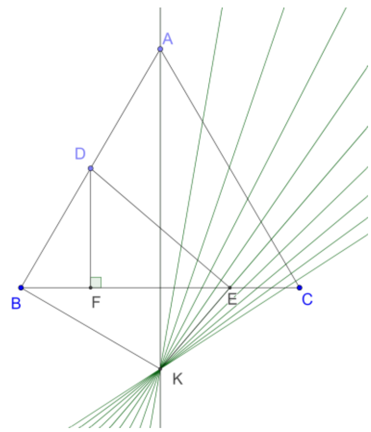

Figure 5. Fixed point of a family of the line for problem 5 in GeoGebra

Proof. $EF = BC/2 = BH$. Hence $BF = BH - FH = EF\text{-}FH\text{=}EH$. $\widehat{EDF} = \widehat{KEH}$. So $\Delta EDF \sim \Delta KEH$. Hence $\frac{DF}{EH} = \frac{ED}{EK}$ or $\frac{DF}{BF} = \frac{DE}{EK}$ $\Rightarrow$ $\Rightarrow \Delta FBD \sim \Delta EKD$ (SAS). Hence $\widehat{BDF} = \widehat{KDE} \Rightarrow \widehat{BDK} = \widehat{EDF}$. But $\widehat{EDF} = \widehat{KEH}$. So B, D, E, and K are concyclic and $\widehat{DBK} = 90^0$. Hence K is a fixed point.

Problem 6: Given circle (O) of diameter AB. On the tangent Ax to (O), select point C such that $AC = AB$. Line BC intersects (O) at D. M is a point moving along line AD. N and P are projections of M onto AB and AC, respectively. H is the projection of N onto PD. Prove that when M moves, HN always goes through a fixed point.

Solution: Draw segment AB and circle with the diameter of AB. Draw tangent Ax of (O). Select C as the intersection of this tangent and the circle (A, AB). D is the intersection of (O) and BC. Select point M on segment AD. Draw the perpendicular lines to find N, P, and H. As M moves along AD, HN always goes through a fixed point E on (O).

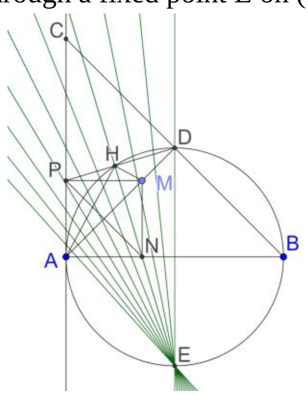

Figure 6. Fixed point of a family of the line for problem 6 in GeoGebra

Proof. Let E be the intersection of HN and (O). ΔABC is an isosceles right triangle (AC=AB). AD is the altitude so it is also the angle bisector. The quadrilateral APMN is a rectangle and AN = MN (triangle AMN is an isosceles right triangle). Hence APMN is a square. As $\widehat{APM} = \widehat{AHM} = \widehat{ANM} = 90^0$so A,P,H,M, and N are concyclic on the circle of diameter. Hence

$$\widehat{AHN} = \widehat{APN} = 45^0, \widehat{MHN} = \widehat{MAN} = 45^0$$

$$that\ implies \Rightarrow \widehat{AHB} = \widehat{AHN} + \widehat{NHM} = 90^o.$$

2.2. Teaching experiment results

Students from two Grade 9 classes at Newton Grammar School (136 Ho Tung Mau, North Tu Liem District, Ha Noi) participated in the experiment. The students had equivalent levels of proficiency in geometry. The control group consisted of 29 students who were taught using traditional methods, while the experimental group included 30 students who were taught how to use GeoGebra to assist in solving problems.

The experiment was conducted in four following stages:

Stage 1: The experimental group was introduced to GeoGebra.

Stage 2: Both classes were taught the basic knowledge from Chapter 2: Angles and Circles in the Grade 9 mathematics textbook.

Stage 3: Students practiced solving problems, including proofs of fixed points for families of lines and circles. In this stage, the experimental group received instruction on using GeoGebra for visual assistance in finding solutions, while the control group followed traditional methods under the teacher's guidance.

Stage 4: Both groups took the same 45-minute test, which included multiple-choice and written problems related to fixed points of families of lines and circles. The experimental group completed the test using GeoGebra in exam mode, where only GeoGebra was available on their devices, and Internet access was restricted.

The results of the experiment are presented in Table 1.

Table 1. Test scores of both groups

Statistics of scores	**Experimental group (30 students)**	**Control group (29 students)**
Mean	7.07	5.60
Median	7	6
Mode	7	5
Standard Deviation	1.64	1.30

T-test results showed significant differences between the experimental and the control group (p-value = 0.001134 < 0.05).

3. Conclusions

Several geometrical problems related to the Vietnamese mathematics curriculum have been explored using GeoGebra by the authors in previous studies [3]-[10]. As demonstrated in Section 2, GeoGebra can be utilized both as a visual demonstration tool and as an explorative method for solving problems involving fixed points. The use of GeoGebra not only enables students to easily visualize concepts but also saves time when finding proofs. Additionally, teachers can use GeoGebra to design interactive activities that encourage students to approach the topic in a more active and creative manner.

Preliminary teaching experiments also indicated that GeoGebra effectively improves students' problem-solving proficiency and enhances their geometrical reasoning skills. Future research could focus on more complex problems involving dynamic geometry to further assess the value of dynamic geometry software in improving mathematics teaching methods within the Vietnamese curriculum.

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