

LANDMARKS AND ORIENTATIONS IN TEACHING AND LEARNING SCHOOL MATHEMATICS

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Received June 22, 2024. Revised September 16, 2024. Accepted December 27, 2024.

Abstract. This talk explores the interconnected and critical issues of competencies, applications, and assessments in the teaching and learning of school mathematics. We examine the dual roles of competencies as both helpful and questionable, the limits and significance of applications, and the unintended consequences of assessments. These themes are intricately linked and profoundly influence one another. To address these challenges effectively, we analyze the core nature and character of school mathematics as a foundation for informed decision-making. Additionally, we consider the transformative and unpredictable impacts of artificial intelligence (AI) on mathematics education, concluding with four practical takeaways for educators and researchers.

Keywords: competencies, mathematics education, artificial intelligence, educational assessment, applications in Mathematics.

1. Introduction

The teaching and learning of school mathematics play a crucial role in developing students' cognitive abilities, fostering problem-solving skills, and preparing them for future challenges. However, significant challenges persist in the field, including fragmented lesson structures, the overemphasis on competencies, and the controversial role of assessments. Heymann (1996) highlighted the necessity of central ideas in mathematics education to prevent lessons from being reduced to disconnected topics and to help students understand the broader structure of mathematics [1]. Similarly, the Danish KOM Project (2002) introduced a competency-based framework, shifting the focus from content to cognitive processes. However, critiques of the model reveal its limitations in portraying mathematics as purely functional rather than preserving its epistemological essence [2]. The increasing integration of applications into mathematics education has further complicated the landscape. While initiatives like the International Study Group for the Teaching of Mathematical Modelling and Applications (ICTMA) emphasize the importance of contextualizing mathematics to real-world scenarios, they often risk overshadowing the discipline's intrinsic power of abstraction and decontextualization [3]. Furthermore, assessment-driven reforms, influenced by global assessments such as PISA, have promoted a culture of examination at the expense of meaningful learning experiences [4]. This study seeks to address these interconnected issues by offering a unified perspective on competencies, applications, and assessments in mathematics education. Unlike previous research, which often isolates these elements, this work highlights their collective impact and the need for

coherence. Additionally, the study explores the transformative potential of artificial intelligence (AI) in personalizing and adapting mathematics teaching and learning, a relatively underexplored domain in existing literature. Through this novel approach, the research aims to provide meaningful orientations for improving mathematics education in an evolving educational landscape.

2. Content

2.1. Main problems

Figures 1 and 2 show the main problems of organizing and teaching mathematics lessons.



Figure 1. a sandheap

1. Vereinfachen Sie die folgenden Terme so weit wie möglich:

a) $8a + 4a$	b) $8a \cdot 4a$	c) $6x - 14x$
d) $6x : 14x$	e) $7x + (4x - 3)$	f) $-14m - (21m + 15)$
g) $6z - (3z - 6) + (7 - 8z)$	h) $3a - [4 - (2a - 7)]$	i) $(-2a)(-3b)$
j) $(-3xy)(-5z)(-2)$	k) $u^5v^4(-uv)^3$	l) $a^2bc^2(-ab^3)^2 \cdot 5ac^3$
m) $-3(a + b)$	n) $5(4a - 7b) + 3a$	o) $(-9)(r - 4s)$
p) $(-6a^3) : (-3a)$	q) $24mn^2 : (-8n^2)$	r) $-\frac{12}{5}ab^3c^2 : (\frac{4}{15}ab^2)$
s) $(26a - 65b) : (-13)$	t) $(8e + 12f - 10g) : 2$	u) $(ax^5 - bx^3) : (-x^3)$
v) $30a : (2a : 5)$	w) $4x \cdot (y \cdot x)$	x) $2(6rs + 4rt)$

2. Vereinfachen Sie die folgenden Terme so weit wie möglich:

a) $2x - 7y - 21z - 13x + 4y - 17z$
 b) $(-3a - b + c) - (6a - 3b + 2c)$
 c) $17x - (43y + 6z) + [-(48x - 53z) + 4y]$
 d) $3(2a - b) - 2(4a + 3b)$
 e) $4(6x - 5) - 2(3x - 7)$
 f) $5a^2(4a^3 - 1) + a^4(a - 2)$
 g) $-3x(5x^2 - 3) + (-2x^2)(5x - 1)$
 h) $[-3(a + b)^2 + 2(a + b)^2] : (a + b)$
 i) $(-\frac{3x^2}{2} + \frac{4x^4}{3} + 2x^3) : (-\frac{7x^2}{2})$

3. Berechnen Sie die folgenden Produkte von Summen:

a) $(x - y)(m + n)$	b) $(t - 2f)(3t + 5f)$	c) $(ty + n)(5ty - 3n)$
d) $(2b + 3)(5c + 8)$	e) $(m^5 - 2)(3 - 2m)$	f) $(ax - b)(3ax + 5b)$

4. Berechnen Sie mithilfe der binomischen Formeln:

a) $(r + s)^2$	b) $(r - s)^2$	c) $(x + y)(x - y)$
d) $(7c - 3d)^2$	e) $(m^5 + 2)^2$	f) $(10u + 12v)(10u - 12v)$
g) $(4x^2 - 3y^2)(4x^2 - 3y^2)$	h) $(a + b)^2 - (a - b)^2$	i) $(3x - 5)^2 - (5 - 3x)^2$
j) $(a^3 - 1)(a^3 + 1)$	k) $3(3u + 2)^2 + (8u - 3)(1 - 4u)$	
l) $3x(5x - 8) - (4x - 3)^2$	m) $a^2z^2 - (az - 1)^2 + 2(3 - az)$	

Figure 2. a page from a German mathematics textbook

Note for Figure 2: Translation of the work instructions: 1. and 2. Simplify the following terms as much as possible. 3. Calculate the following products of sums. 4. Calculate using the binomial formulas.

- A) What is the core or essence of this substance 'school mathematics'?

B) School mathematics should not look like a sand heap neither overall or in a single lesson. They need a recognizable structure on the whole and in detail.

- We must discuss which kinds of cognitive activities we want to activate and foster in the teaching-learning process in mathematics lessons.

The sandheap substance problem 1. As we discuss later below.

The sandheap structure problem 1. B was raised and discussed by Hans Werner Heymann in his famous and epochal book 'Allgemeinbildung und Mathematik' (Weinheim und Basel 1996). (A lightly shortened version for the English reader: Hans Werner Heymann: Why Teach Mathematics: A Focus on General Education. Translated by Thomas La Presti. Mathematics Education Library. Dordrecht (NL): Kluwer 2003). In chapter 4.2.3 he discusses three motives for a 'central ideas' concept:

- (1) The aim is to prevent the phenomenon so often complained of, that mathematics lessons are broken down into isolated items.
- (2) On the basis of central ideas, the pupils should be able to gain an appropriate picture of mathematics and be able to recognize the "structure of the subject".
- (3) With the help of central ideas, it should become visible to the learners how the mathematics taught relates to the rest of the world that the students can experience".

After compiling a synopsis: and catalogs of central ideas for mathematics education (from the second half of the 20th century), he proposes his catalog:

- Idea of Number;
- Idea of Measurement;
- Idea of Spatial Structuring;
- Idea of Functional Coherence;
- Idea of the Algorithmus;
- Idea of Mathematical Modelling.

Certainly, today we miss the Idea of Uncertainty or something like that, but despite the compatibility with Bruner's much-propagated conception of a spiral curriculum, which requires the realization of such central ideas, there was no real further development or sharpening of the 'Big Ideas' (Leading Ideas/Strands/Domains). The reason for this deficiency is the concentration of the discussion on the second problem, the tenor of which ultimately pushes the material and content of school mathematics completely aside.

2.2. Competence and competencies

The cognitive activities problem (2. B) was treated in a groundbreaking and momentous way in The Danish KOM project (2002) under the direction of Mogens Niss [2]:

"The fundamental idea of the project is to base the description of mathematics curricula primarily on the notion of a "mathematical competency", rather than on syllabi in the traditional sense of lists of topics, concepts, and results. This allows for an overarching conceptual framework which captures the perspectives of mathematics teaching and learning at whichever educational level".

"There are eight competencies which can be said to form two groups. The first group of competencies are to do with the ability to ask and answer questions in and with mathematics:

1. Thinking mathematically;
2. Posing and solving mathematical problems;
3. Modelling mathematically;
4. Reasoning mathematically.

The other group of competencies are to do with the ability to deal with and manage mathematical language and tools:

5. Representing mathematical entities;
6. Handling mathematical symbols and formalisms;
7. Communicating in, with, and about mathematics;
8. Making use of aids and tools.

In the worldwide adaptation, adoption, and transformation of these competences, it has been and is overlooked that they are based on a picture of the nature of mathematics and school mathematics, which, by the way, we will also encounter later in the PISA framework.

“To master mathematics means to possess mathematical competence. But then, what is that? To possess a competence (to be competent) in some domain of personal, professional, or social life is to master (to a fair degree, modulo the conditions and circumstances) essential aspects of life in that domain. Mathematical competence then means the ability to understand, judge, do, and use mathematics in a variety of intra- and extra-mathematical contexts and situations in which mathematics plays or could play a role”.

Mastering mathematics means here to apply and use mathematics. Mathematics is not anymore a domain or matter in its own right, an epistemological approach to the world, but a tool for dealing with it. The functional character of mathematics in this view is most evident when mathematics is replaced by music or another field of education in the above quote. The concept of general education is here narrowed down to a social education of utility. While this does not deprive the eight competencies of their meaning or significance, it does shed light on their tenor and background, and also explains the nature of their reception.

Furthermore, the grey ‘Competence Flower’ (Figure 3) comes down to a detailed matrix of everything which in a nonsensical way parcels out the cognitive activities in the teaching and learning of mathematics (Figure 4).

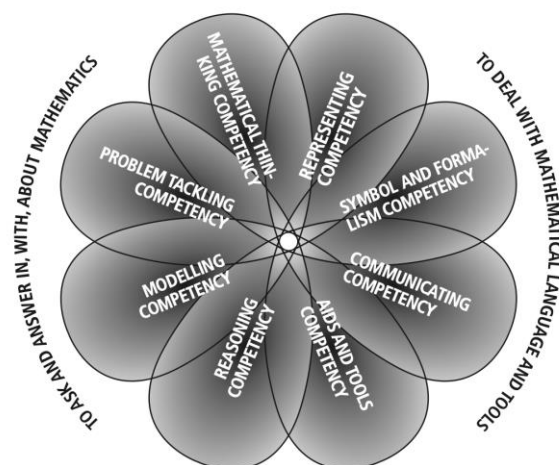


Figure 3. The Competence Flower

Subj. matter area Competency	Area 1	Area 2	...	Area n
Mathematical thinking				
Problem handling				
Modelling				
Reasoning				
Representation				
Symbols and formalism				
Communication				
Aids and tools				

Figure 4. The separating table of everything from Area 1 to Area n

In the article ‘Mathematical competencies revisited’, Mogens Niss and Tomas Højgaard have given the 2002 concept of competence an additional orientation [5]:

“Competence is someone’s insightful readiness to act appropriately in response to the challenges of given situations”.

“Mathematical competence is someone’s insightful readiness to act appropriately in response to all kinds of mathematical challenges pertaining to given situations”.

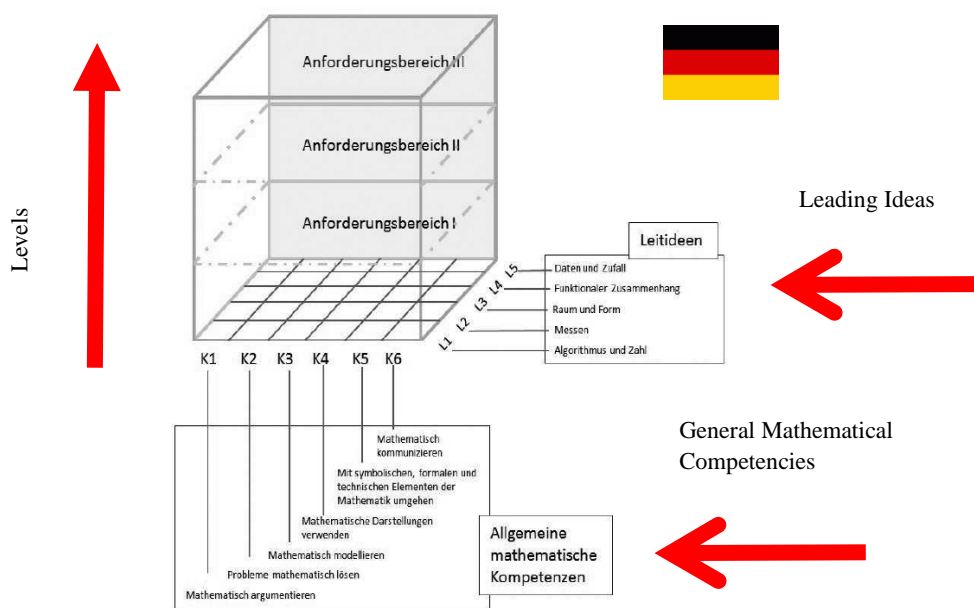


Figure 5. Three-dimensional competence matrix for Upper Secondary in Germany

But this extension of attaching to the usefulness of mathematical education an unmotivated and poorly explained 'readiness to act' was hardly heard in the international discussion. This made the national adoption of Table 4 all the more effective, for example in Germany. For the Upper Secondary, the eight competencies were reduced to six, five areas were named, and three levels were introduced, so that a three-dimensional matrix with a total of 90 competencies was created (Figure 5).

Austria followed the same scheme with four competencies and four content areas at three levels, thus establishing 48 competence monads, which were carefully labeled and explained (Figure 6).

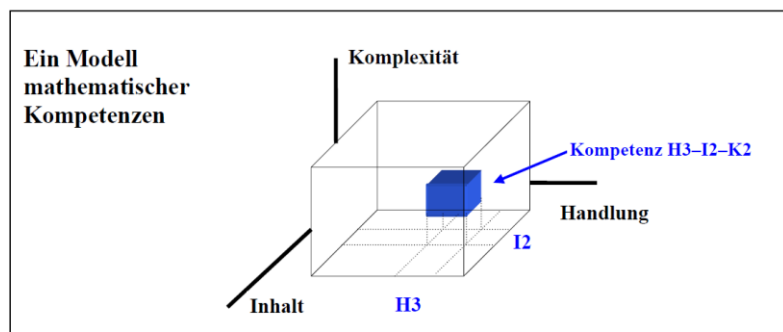


Figure 6. The 3D-Modell of mathematical competencies in Austria with the dimensions of content action and complexity

Even if such competence models are neither suitable for theoretical analysis nor for practical application, they are nevertheless propagated – for example, in the so-called 'Bildungsstandards in Mathematik', which were adopted by the Ministers of Education and Cultural Affairs of the German Länder in 2012. The compound 'Bildungsstandards' is a strange and contradictory amalgamation of the German term 'Bildung', which can be interpreted in very different and multifaceted ways against the background of German educational philosophy, with the Anglo-Saxon term 'standards', which was initially coined mainly for the uniformity of industrial products in the course of quality control. Behind this term today lies a parastatal didactics that is imposed on teachers with the illusory aim of increasing the quality of education, which in turn is to be measured by questionable tests.

Thus, the meaningful and valuable discussion of the cognitive activities associated with teaching and learning and teaching mathematics ended in an official disaster that did not advance the students, the teachers, or the teaching of mathematics.

2.3. Applications - Why (not)

To clarify the terms, we would first like to point out that 'applications' can be viewed from different perspectives. You can use mathematics to understand the world, and you can use the world to understand mathematics. The former can be related to terms such as modeling or application, the latter to terms such as emergence (Freudenthal) [4], visualization, illustration, and dressing or clothing. Confusion of these points of view leads to misunderstandings and, by the way, usually to poor, unpalatable application tasks.

However, after a period - starting at the end of the Second World War - to regard mathematics primarily as pure mathematics and to teach it in colleges and schools, which can also be traced back to the undertaking Bourbaki, among others, there was an increasingly strong development

to demand the inclusion of applications in mathematics education. One example is the creation of the ICTMA group.

“ICTMA, the International Study Group for the Teaching of Mathematical Modelling and Applications, has been an Affiliated Study Group of the International Commission on Mathematical Instruction (ICMI) since 2003. ICTMA emerged from the International Community of Teachers of Mathematical Modelling and Applications (hence the acronym ICTMA), a community of scholars and researchers supporting the series of International Conferences on the Teaching of Mathematical Modelling and Applications and which has been concerned with research, teaching, and practice of mathematical modeling since 1983. The group aims to foster the teaching and researching of the teaching and learning of mathematical modeling and the ability to apply mathematics to genuinely real world problems in primary, secondary, and tertiary educational environments including teacher education, together with professional and workplace environments” [6].

As a result of this development, which was initially welcome, mathematics and its applications were increasingly merged, especially in school lessons, as if they were the essential reason and central motivation for practicing and teaching mathematics in the first place. The educational value of mathematics lies solely in its usefulness. This functional view and conception of education is essentially propagated and promoted by organizations such as the OECD, which, through its PISA serial international comparative survey, virtually prescribes to the participating countries that the usefulness of mathematics is their only *raison d'être* [7]. The result is textbooks with titles such as *Mathematics in Context* and a plethora of text-heavy and artificial tasks in which mathematics almost disappears behind the context under consideration. It is completely ignored here that the universal power of mathematical thinking is based on decontextualization and not on embedding in real contexts.

All in all, therefore, the attempt to enrich school mathematics with applications has overshoot this sensible goal by far and thus destroys the construction of an appropriate view of this discipline instead of promoting it.

2.4. Assessment

I would like to make a few brief remarks about assessment, which seem so necessary, among other things, because today this topic is mainly associated with international comparative studies such as PISA.

The design, nature, and content of exams determine and shape the teaching that precedes them, what I call prefiguration, which should not be underestimated in its impact. Especially in the final classes, meaningful lessons are often replaced by test preparation. While an open learning culture is usually to be strived for, this is now turning into an examination culture that aims solely at performance. A pure waste of time and manpower. Either we change the learning and teaching culture, or we change the examen culture in a way that reflects our teaching and education aims.

2.5. The nature of Mathematics

In their 2019 article, Mogens Niss and Tomas Højgaard state [5]:

“As a scientific discipline mathematics shares some properties with other disciplines whereas several other significant properties are peculiar and specific to mathematics, in particular the ways mathematics obtains and justifies its results. What exactly are the properties that mathematics has in common with other fields and disciplines, what are the properties that are peculiar to mathematics, and what are the causes for the similarities as well as for the differences?”.

For about two decades, there has been a broader discussion about the nature of science, which on the one hand tries to characterize the natural sciences and on the other hand calls for such

characterizations to be explicitly made the subject of school teaching. Analogous considerations of mathematics are missing. Instead, in PISA, for example, the term mathematical literacy propagates a problematic and questionable conception of mathematical education, which functionalizes it and restricts it to familiarizing oneself with algorithms useful for daily and civic life [8].

“Mathematical literacy is an individual’s capacity to reason mathematically and to formulate, employ, and interpret mathematics to solve problems in a variety of real-world contexts. It includes concepts, procedures, facts, and tools to describe, explain, and predict phenomena. It helps individuals know the role that mathematics plays in the world and make the well-founded judgments and decisions needed by constructive, engaged, and reflective 21st Century citizens”.

Mathematics as a discipline that argues, justifies, and proves, as its approach to the world, as a formal theory, is fundamentally ignored here. This is also evidenced by the choice and content of the items in the international performance comparison studies.

For a meaningful foundation of school mathematics, from which statements can be derived about the desired cognitive activities, the role of applications and the forms of assessment, a debate about the nature of mathematics and school mathematics is needed instead of a definition and review of industrial performance norms and so-called standards by commercial education providers.

2.6. Epilog and Prolog: Artificial Intelligence and the Future of Maths Education

Thomas: Why do we learn something?

ChatGPT: We learn for various reasons, including personal growth, acquiring skills for employment, understanding the world, and enhancing problem-solving abilities. Education serves both practical and intellectual purposes in our lives.

Thomas: What is the future of maths lessons facing AI?

ChatGPT: The future of math lessons with AI involves personalized learning, adaptive tutoring, and interactive simulations. AI can provide tailored exercises, instant feedback, and individualized challenges, fostering a more engaging and efficient learning experience. However, human guidance remains crucial for deeper understanding and contextual application of mathematical concepts.

As is to be expected mathematically with an averaging, CHATGPT formulates a mixed miscellaneous.

3. Conclusions

This study emphasizes the critical need for a balanced approach in mathematics education by integrating competencies, applications, and assessments without letting any single aspect dominate. By fostering competencies, realistic applications, and meaningful assessments, mathematics lessons can better prepare students for real-world challenges while preserving the intrinsic nature of the discipline. However, the findings caution against competence-, application, or assessment-orientated teaching, as such approaches risk undermining the coherence and depth of mathematics education. Future research could explore innovative teaching strategies that align with this balanced framework, particularly in light of technological advancements like artificial intelligence. Investigating the interplay between AI-driven personalization and the fundamental nature of mathematics could lead to groundbreaking insights for both theoretical and practical applications in education. By focusing on these directions, the field can continue to evolve while staying true to the essence of mathematics.

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